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m-n COMPACTNESS and CARDINAL INVARIANTS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled m-n COMPACTNESS and CARDINAL INVARIANTS submitted by U.N.B. DISSANAYAKE in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

ABSTRACT

In the first part of the thesis we investigate the productivity of weak m-n compactness which covers three main concepts: H(i), feebly compact and weakly Lindelöf. In the later part, we consider the possible improvements and generalizations of Arhangel'skii's famous theorem (1970),

$$|X| \leq 2^{L(X)\chi(X)}$$

for T_2 spaces, where $L(X)$ is the Lindelöf degree and $\chi(X)$ is the character of the space X.

Chapter I contains a brief history of cardinal invariants in topology and definitions of generalized topological notions with necessary basic theorems.

Chapter II is devoted to a short survey of m-n compact topological spaces and considers the problem of preservation of tightness in large products of m-n compact spaces. In this respect we show that tightness behaves similar to character in products of locally Lindelöf, T_3 P-spaces and hence in products of Lindelöf, T_3 P-spaces.

In Chapter III, we introduce strong m-n filters and study products of two spaces and then large products (and weak topological sums). By doing this we obtain sufficient conditions on factor spaces to ensure the productivity of weak m-n compactness in arbitrary products.

Chapter IV is mainly devoted to a study of two interesting cardinal functions, the Lindelöf degree, $L(X)$ and the weak Lindelöf number, $wL(X)$. Here we introduce a new cardinal function, the almost Lindelöf degree $aL(X)$, which satisfies,

$$wL(X) \leq aL(X) \leq L(X)$$

for any space X and prove that

$$|X| \leq 2^{aL(X)\chi(X)}$$

for T_2 spaces. Next, we obtain results similar to the above involving $wL(X)$ and study relations between $L(X)$ and $wL(X)$ by introducing a new cardinal function, the turf number.

INTRODUCTION

In 1929, P.S. Alexandroff and P.S. Uryshon introduced H-closed spaces as a natural generalization of compact, T_2 spaces. In 1941, C. Chevalley and O. Frink proved that an arbitrary product of H-closed spaces is H-closed. Z. Frolik, W. Comfort, A Hager and S. Negrepontis have studied another generalization of compact (Lindelöf) spaces, namely weakly Lindelöf spaces. In 1972, M. Ulmer [69] has extended some of the results of the above mathematicians, by proving that the weakly Lindelöf property in a product space is determined by finite subproducts. The above two variations of compactness are special cases of the general concept weak m-n compactness and part of our interest is to find sufficient conditions on small subproducts (and individual factor spaces) to preserve weak m-n compactness in arbitrary products.

In 1923, P.S. Alexandroff raised a question which subsequently received a great deal of attention:

> Does every first countable bicompactum (compact, T_2) have cardinality at most $\mathfrak{c}$?

Nearly fifty years later A.V. Arhangel'skii answered this question by proving that if X is a T_2 space, then

$$|X| \leq 2^{L(X)\chi(X)}, \tag{A}$$

where $L(X)$ is the Lindelöf degree and $\chi(X)$ is the character of the space X.

After this theorem, cardinal functions in topology played an active role. I. Juhász has written a definitive text (see [34]) in this field, including all the new cardinal functions and new results. As a natural generalization of $L(X)$, the weak Lindelöf number $wL(X)$ ($\leq L(X)$) was first introduced in [4] and there it is proved that, if X is a T_4 space, then

$$|X| \leq 2^{wL(X)\chi(X)} \tag{B}$$

The remaining part of our interest is to study the cardinal function $wL(X)$ in contrast to $L(X)$ and to consider possible improvements of (A) and to extend (B) to the class of T_3 spaces.

Chapter I is devoted to a brief study of basic topological notions, namely regular cardinals, cardinal invariants ($\chi(X)$, $\psi(X)$, $w(X)$, $L(X)$ etc.), filters and products.

In Chapter II, our main aim is to estimate tightness in products of m-n compact spaces via the factor spaces. We begin this by giving a short survey of the theory of m-n compact spaces and proving a product theorem which partially extends Noble's theorem 4.2 of [53]. That product theorem, and the resulting theorem on tightness, are described below.

We call X a quasi n-paracompact space if and only if every open cover of X has a $<n$-closure preserving refinement, where $n \geq \aleph_1$.

(1) Let $X = \Pi\{X_i : i \in I\}$ be a quasi n-paracompact space. Let n be a regular cardinal and suppose each X_i is ∞-n compact, <n-discrete and $\dot{T}_3$. Then X is ∞-n compact.

Let $X = \Pi\{X_i : i \in I\}$. Let $\partial_I(X) = \sup\{\partial(X_i) : i \in I\}$ where $\partial(X)$ denotes the tightness of X. We define X to be a GLC (n,i)-space for i = 1,2,3,4 if X is a locally ∞-n compact, <n-discrete, T_i space and show that, if each $X_i \in$ GLC (n,3), then

(2) $$\partial(X) = |I| \cdot \partial_I(X),$$

generalizing 5.9 in [34].

In Chapter III we study the productivity of weak m-n compactness and prove the following:

(3) Let $X = \Pi\{X_i : i \in I\}$ and let $n > \gamma \geq k \geq \aleph_0$. Suppose n is regular and strongly γ-inaccessible; then $(\Pi X_i)_k$ (ΠX_i with k-box topology) is weakly ∞-n compact if and only if the sub-products $(X_{I'})_k$ are weakly ∞-n compact for all $I' \in P_\gamma(I)$ where $P_\gamma(I) = \{I' \subset I : |I'| < \gamma\}$

The special case $\gamma = k = \aleph_0$ of (3) produces 1.3 of [69].

We also show that in the presence of <n-discreteness or local compactness, an arbitrary product of weakly ∞-n compact spaces is weakly ∞-n compact, where n is a regular cardinal.

In Chapter IV, we introduce a new cardinal function, the almost Lindelöf degree, $aL(X)$, which agrees with $L(X)$ on T_3 spaces but which is often smaller than $L(X)$ on T_2 spaces, and prove that

$$(4) \qquad |X| \leq 2^{aL(X)\chi(X)}$$

for T_2 spaces.

Next, we introduce, via local π-bases, a new class of spaces, the Π-normal spaces. This class contains T_4 spaces and

$$|X| \leq 2^{wL(X)\chi(X)}$$

when X is a Π-normal space. This extends (B).

Since compact subsets behave nicely in T_2 spaces, it is of interest to obtain upper bounds for the number of compact subsets $\mathcal{K}(X)$ of X. In this direction, we show that for a T_2 space X,

$$(5) \qquad |\mathcal{K}(X)| \leq 2^{aL^*(X)\bar{\psi}(X)} \leq \min\left(2^{L^*(X)}, 2^{2^{d(X)}}\right)$$

(see Section 1 in [9]).

Finally, we introduce a new cardinal function, the turf number $T(X)$, and show that, for regular spaces,

$$(6) \qquad L(X) \leq T(X)wL(X).$$

Each chapter is divided into sections and subsections. The main results in each subsection are labelled by letters A,B,C,D,E etc. When we quote a result in the same subsection we use only a letter and if the result which we quote is in the same section but in a different

subsection, then we use only the subsection number followed by the corresponding letter. We follow this pattern by indicating chapter number, section number, subsection number and the corresponding letter, as necessary.

The cardinal functions which are not defined here can be found in Juhász text [34] and for topological concepts which are used without any special introduction, we direct the reader to the text of Willard [80].

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CHAPTER I: PRELIMINARIES

1. Cardinal Arithmetic

This section is devoted to a brief review of transfinite arithmetic and some special properties of infinite cardinals. For notation and terminology not explained here, as well as a more careful development of results quoted here, we refer the reader to [30].

A. Simple Rules. Let α, β and γ be any three cardinals. Then

(i) $(\alpha^{\beta})^{\gamma} = \alpha^{\beta\cdot\gamma}$,

(ii) $|P(A)| = 2^{\alpha}$, where $|A| = \alpha$,

(iii) $\alpha^{\beta} \leq \alpha^{\gamma}$, whenever $\beta < \gamma$.

B. Theorem. Let α and β be any two cardinal numbers such that the larger is infinite and the smaller is non-zero. Then

$$\alpha + \beta = \alpha \cdot \beta = \max \cdot (\alpha, \beta).$$

This is very fundamental and it is called the Absorption Law of cardinal arithmetic.

If α is an infinite cardinal number, then α^{+} denotes its successor, which exists because $\alpha < 2^{\alpha}$ and the set of all cardinals less than or equal to 2^{α} is well-ordered.

Next, we shall look into the properties of infinite cardinals. First we need a definition.

C. Definition. Let $(X,<)$ be a linearly ordered set. Then a subset A of X is said to be cofinal in X if for every $x \in X$, there exists $a \in A$ such that $x \leq a$.

Conventionally we identify an infinite cardinal α with the least ordinal κ_α with cardinality α. Thus we can define the cofinality of α as,

$$cf(\alpha) = \min \{|A| : A \subset \kappa_\alpha \text{ and } A \text{ is cofinal in } \kappa_\alpha\}.$$

Since α itself is cofinal in α, $cf(\alpha) \leq \alpha$.

D. Facts About $cf(\alpha)$. (See [30].)

(i) $cf(\alpha) = cf(cf(\alpha))$

(ii) $cf(\alpha^+) = \alpha^+$

(iii) $\alpha^{cf(\alpha)} > \alpha$

(iv) $cf(2^\alpha) > \alpha$

E. Definition. A cardinal number k is said to be regular if and only if $k = cf(k)$ and it is said to be singular if and only if $k > cf(k)$.

The cardinal $\aleph_0$ is the cardinality of any set which is equipotent to $\mathbb{N}$ and $\aleph_1$ is its successor and so on. Assuming the continuum hypothesis, $\aleph_1 = 2^{\aleph_0}$ $(= c)$.

All finite cardinals are singular, while $\aleph_0, \aleph_1, \ldots \aleph_k, \ldots,$ $k < \omega$, are regular cardinals. The cardinal

$$\aleph_\omega = \sup \{\aleph_0, \aleph_1, \ldots \aleph_k, \ldots\}$$

is a singular cardinal.

We note that, α^+ is always a regular cardinal for any cardinal α.

The cofinality of an infinite cardinal α can be regarded as the least cardinal β such that α can be decomposed into union of β sets, each of which has cardinality less than α. As an example, $\aleph_\omega = \bigcup_{k=0}^{\infty} \aleph_k$. Thus $cf(\aleph_\omega) = \aleph_0$.

Next, we shall study special properties of cardinals.

F. Definition. A cardinal α is said to be strongly inaccessible if and only if,

(i) $\alpha > \aleph_0$, and

(ii) $2^\beta < \alpha$ for all $\beta < \alpha$, and

(iii) α is regular.

A cardinal which satisfies (ii) and (iii) is called an inaccessible cardinal and a cardinal which satisfies (ii) is called a strong limit cardinal.

We shall specialize further. Let α and γ be cardinals such that $\gamma \leq \alpha$. Then α is said to be strongly γ-inaccessible if $\beta^k < \alpha$ whenever $\beta < \alpha$ and $k < \gamma$. Note that every infinite cardinal is strongly $\aleph_0$-inaccessible.

We shall next turn to infinite sums and products.

G. Definition. Let $\{k_i : i \in I\}$ be a set of cardinal numbers. Let $\{X_i : i \in I\}$ be a disjoint family of sets such that $|X_i| = k_i$ for

each i. Then we define

$$\sum\{k_i : i \in I\} = |\cup\{X_i : i \in I\}|,$$

and

$$\Pi\{k_i : i \in I\} = |\Pi\{X_i : i \in I\}|.$$

The following results will be useful in the sequel (see [30]):

(i) If λ is an infinite cardinal and $k_i > 0$ for each $i < \lambda$, then

$$\sum\{k_i : i < \lambda\} = \lambda \cdot \sup\{k_i : i < \lambda\}.$$

(ii) If λ is an infinite cardinal and $\{k_i : i < \lambda\}$ is a non-decreasing sequence of non-zero cardinals, then

$$\Pi\{k_i : i < \lambda\} = (\sup\{k_i : i < \lambda\})^{\lambda}.$$

Finally, we shall introduce some notation which will be useful in later work.

H. Notation.

(i) Let α, β and γ be infinite cardinals. Then $\alpha^{\underline{\beta}}$ denotes the sum,

$$\sum\{\alpha^{\gamma} : \gamma < \beta\}.$$

(ii) Let α be an infinite cardinal. Then $\log \alpha$ denotes the

$$\min\{\beta : 2^{\beta} \geq \alpha\}.$$

To illustrate, note that

(i) If $\alpha \geq \aleph_0$, then

$$\alpha^{\underline{\aleph}_0} = \alpha \quad \text{and} \quad (\alpha)^{\underline{\alpha}^{+}} = 2^{\alpha}$$

(ii) $\log c = \aleph_0$.

2. Cardinal Invariants

The development and use of cardinal invariants in topology has been significant. At early stages of topology, character $(\chi(X))$, pseudocharacter $(\psi(X))$ and weight $(W(X))$ played an important role. After 1970, other cardinal invariants such as the Lindelöf degree $(L(X))$, density $(d(X))$ and spread $(s(X))$ have taken an active role. As we shall see, many valuable results can be formulated in terms of cardinal invariants.

A. Basic Definitions. Let X be any topological space and let $a \in X$. Then an open neighbourhood base at a is a collection $\mathcal{V}_a$ of open subsets of X such that if V is any open subset containing a, then there exists some $u \in \mathcal{V}_a$ such that $a \in U \subseteq V$.

If X is a T_1-space, then a pseudobase at a is a collection $\mathcal{W}_a$ of open subsets of X such that

$$\{a\} = \cap \{W : W \in \mathcal{W}_a\}$$

(i) Let X be any topological space and let $x \in X$. Then we define, $\chi(x,X) = \min\{|\mathcal{V}_x| \; \mathcal{V}_x$ is an open neighbourhood base at x and the character $\chi(X)$ of X as,

$$\chi(X) = \sup\{\chi(x,X) : x \in X\}$$

(ii) Let X be a T_1-space and let $x \in X$. Then we define $\psi(x, X) = \min\{|\mathcal{W}_x| : \mathcal{W}_x$ is a pseudobase at $x\}$ and the pseudocharacter $\psi(X)$ of X as,

$$\psi(X) = \sup\{\psi(x,X) : x \in X\}.$$

(iii) Let X be any topological space. Then we define the <u>weight</u> $w(X)$ of X as,

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \aleph_0.$$

(iv) Let X be any topological space. Then we say that X is <u>k-Lindelöf</u> if and only if every open cover of X has a sub-cover of cardinality at most k. We define the <u>Lindelöf degree</u> $L(X)$ of X as,

$$L(X) = \min\{k : X \text{ is k-Lindelöf}\} + \aleph_0.$$

(v) Let X be any topological space. Then we define the <u>density</u> $d(X)$ of X as,

$$d(X) = \min\{|S| : S \text{ is dense in } X\} + \aleph_0.$$

(vi) Let X be any topological space. Then we define the <u>spread</u> $s(X)$ of X as,

$$s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\} + \aleph_0.$$

B. <u>Historical Facts</u>. We know that for any topological space X, $\psi(X) \leq \chi(X)$. In general, the gap between $\psi(X)$ and $\chi(X)$ is very large. In the early part of the twentieth century, P. Alexandroff and P. Urysohn, proved the following remarkable results:

(i) <u>Theorem</u>. Let X be a countably compact, T_3-space. If every point of X is a G_δ-point $(\psi(X) = \aleph_0)$, then X is a first countable

space $(\chi(X) = \aleph_0)$.

(ii) Theorem. Let X be a compact T_2-space. Then $\chi(X) = \psi(X)$. We note that this result can be extended to locally compact T_2-spaces. Thus the weight $w(X) \leq |X|$ for all locally compact T_2-spaces.

Among all separation axioms, normality plays an important role. F.B. Jones (1937) has proved an interesting lemma concerning normal spaces. To state the lemma using cardinal invariants we require a definition.

Definition. Let X be any topological space. Then we define the closed spread $p(X)$ of X as,

$$p(X) = \sup\{|B| : B \text{ is a closed discrete subspace of } X\} + \aleph_0.$$

(iii) Lemma (Jones). If X is normal, then $p(X) < 2^{d(X)}$. We apply this lemma to the Moore plane M. For this we note that $p(M) = c$ and $d(M) = \aleph_0$. Thus M is a non-normal (completely regular) T_2-space.

Actual development of cardinal invariants, has begun after 1970. This is mainly due to the famous theorem of A.V. Arhangel'skii which answered a long standing problem of P. Alexandroff and P. Urysohn, namely does every first countable, compact T_2-space have cardinality at most c? Arhangel'skii's answer takes the following form:

(iv) Theorem (Arhangel'skii). Let X be a T_2-space. Then

$$|X| \leq 2^{\chi(X)L(X)}.$$

In the latter part of 1970, the development of cardinal functions in topology has proceeded rapidly. As a result of this many new cardinal functions have been added. Among them we shall consider mainly, the following cardinal functions in addition to $\chi(X)$, $\psi(X)$, $W(X)$, $L(X)$, $d(X)$ and $s(X)$, (see [34]):

(a) tightness - $\partial(X)$, (traditional notation)

(b) π-character - $\pi\chi(X)$,

(c) closed spread - $p(X)$,

(d) cellularity - $c(X)$.

C. Examples. The following standard examples are encountered frequently in our work and it is very useful to know the values of some of the cardinal functions on these spaces:

(a) the countable complement topology on an uncountable set X with cardinality k,

(b) the Alexandroff extension of Q (the space of rationals),

(c) the Moore plane,

(d) the Alexandroff double of the unit interval,

(e) the Ordinal space,

(f) the Tychonoff plank,

(g) Michael's line (also known as the scattered line),

(h) the Sorgenfrey line,

(i) the Sorgenfrey plane,

(j) the space $\mathbb{N}^k$ $(k > \aleph_0)$.

D. Chart. We shall give the values of χ, ψ, ∂, $\pi\chi$, d, w, p, L and s for the spaces (a) to (j) in a tabular form.

Let $\phi(X)$ be a cardinal function on a topological space X. Then $\phi^*(X)$ denotes, $\sup\{\phi(Y) : Y \subseteq X\}$. If $\phi = \phi^*$, then we say that ϕ is monotone.

We note that, χ, ψ, ∂, and w are monotone.

SPACES	$\chi(X)$	$\psi(X)$	$\partial(X)$	$\pi\chi(X)$	$d(X)$	$c(X)$	$w(X)$	$p(X)$	$L(X)$	$s(X)$
(a)	k	k	1	k	$\aleph_1$	$\aleph_0$	k	$\aleph_0$	$\aleph_0$	$\aleph_0$
(b)	c	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	$\aleph_0$	$\aleph_0$	$\aleph_0$
(c)	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	c	c
(d)	$\aleph_0$	$\aleph_0$	$\aleph_0$	1	c	c	c	$\aleph_0$	$\aleph_0$	c
(e)	c	c	c	$\aleph_0$	c	c	c	$\aleph_0$	$\aleph_0$	c
(f)	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	c	c	c	c	c
(g)	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	c	c	$\aleph_0$	c	c
(h)	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	$\aleph_0$	$\aleph_0$	$\aleph_0$
(i)	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	c	c	c	c
(j)	k	k	k	k	log k	$\aleph_0$	k	$\aleph_0$	k	k

Next, we shall give an important result which we shall use frequently in our work.

E. Theorem (2.5 of [34]). Let X be a T_2-space. Then $|X| \leq d(X)^{\chi(X)}$.

F. Note. It is easy to construct an example via $\mathbb{N}$, to show that T_2 cannot be relaxed to T_1 in the hypothesis of the theorem E.

G. Example. Let $X = [0,1]^c$. Then X is T_4 and $|X| = 2^c$; $d(X) = \aleph_0$ and $\chi(X) = c$.

Thus the positions of $d(X)$ and $\chi(X)$ are not interchangeable in the right side of $|X| \leq d(X)^{\chi(X)}$.

3. m-n Filters

In the theory of convergence and compactness, filters play an important role. We know that open filters are used in the study of H(i)-spaces (see [62]) and Z-filters are used in the study of the Stone-Cech compactification (see [19]). We shall define m-n filters, which were first introduced by J.E. Vaughan [73].

A. Definition. A collection $\mathcal{F}$ of subsets of a set X has the <n-intersection property if and only if for each $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < n$, $\cap\mathcal{F}' \neq \phi$.

A collection $\mathcal{F}$ of non-empty subsets of a set X is said to be <n-stable if and only if for each $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < n$, there exists a $F \in \mathcal{F}$ such that $\cap\mathcal{F}' \supseteq F$.

B. Definition. A filter $\mathcal{F}$ on a set X is a non-empty collection of non-empty subsets of X which is closed under finite intersections and the superset property.

A sub-collection $\mathcal{F}_B$ of a filter $\mathcal{F}$ is called a base for $\mathcal{F}$ if and only if, $\mathcal{F}$ can be recovered as the collection of all sets which contain some element of $\mathcal{F}_B$.

C. Definition. An m-n filter $\mathcal{F}$ on a set X is a filter on X which has the <n-intersection property and has a base $\mathcal{F}_B$ of cardinality less than or equal to m.

An m-n stable filter $\mathcal{F}$ on a set X is a filter on X which is <n-stable and has a base $\mathcal{F}_B$ of cardinality less than or

equal to m.

D. Examples.

(i) Let X be any first countable topological space. Let $\mathcal{V}_x$ denote the neighbourhood system at $x \in X$. Then, in general, for any infinite cardinals m and n, $\mathcal{V}_x$ is a fixed m-n filter.

(ii) Let X be an infinite set with $|X| = m$ and let Y be an infinite subset of X with $|Y| = n$. Suppose n is a regular cardinal, then

$$\mathcal{F}_B = \{A \subset Y : |Y - A| < \aleph_0\}$$

is a filter base and the filter $\mathcal{F}$ generated by $\mathcal{F}_B$ is a free m-n filter.

In the example (i) if $\{x\}$ is a non-isolated point, then $\mathcal{V}_x$ is not in general $<n$-stable for any $n > \aleph_0$. Next we shall show that any m-n filter $\mathcal{F}$ can be embedded in an m-n stable filter $\mathcal{G}$, under suitable conditions on m and n.

First, we need a convenient notation. For filter bases $\mathcal{F}_B$ and $\mathcal{G}_B$ we write $\mathcal{G}_B > \mathcal{F}_B$ if and only if the filter generated by $\mathcal{G}_B$ contains the filter generated by $\mathcal{F}_B$.

E. Lemma. Let $\mathcal{F}$ be an m-n filter on X and suppose $m^{\underline{n}} = m$ and n is regular. Then there exists an m-n stable filter $\mathcal{G}$ on X such that $\mathcal{G} \supseteq \mathcal{F}$.

Proof. Let $\mathcal{F}_B$ denote a filter base for $\mathcal{F}$ such that $|\mathcal{F}_B| \leq m$ and

$\mathcal{F}_B$ has the $<n$-intersection property.

Set $\mathcal{G}_B = \{\cap\mathcal{F}' : \mathcal{F}' \subset \mathcal{F}_B,\ |\mathcal{F}'| < n\}$.

Then,
$$
\begin{aligned}
|\mathcal{G}_B| &\leq \textstyle\sum\{m^{\alpha} \cdot \alpha < n\} \\
&= m^{\underline{n}} \\
&= m
\end{aligned}
$$

and since n is regular, $\mathcal{G}_B$ is $<n$-stable and clearly, $\mathcal{G}_B > \mathcal{F}_B$. The filter $\mathcal{G}$ generated by $\mathcal{G}_B$ is an m-n stable filter and $\mathcal{G} \supseteq \mathcal{F}$.

F. <u>Notation</u>. Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on a set X. Then $\mathcal{F} \vee \mathcal{G}$ denotes the collection of all finite intersections of members of $\mathcal{F} \cup \mathcal{G}$.

If every member of $\mathcal{F}$ meets every member of $\mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ is a filter base and the filter generated by $\mathcal{F} \vee \mathcal{G}$ is finer than $\mathcal{F}$ and $\mathcal{G}$.

4. Generalized Products

We shall out-line the main facts about the k-box topology on the product $\Pi\{X_i : i \in I\}$. This is a generalization of the Tychonoff product topology, where k is an infinite cardinal number (see [12]).

A. Definition. Let $W = \Pi\{W_i : i \in I\}$ where W_i is a subset of X_i for each $i \in I$. Then the range of W is defined as

$$R(W) = \{i \in I : W_i \neq X_i\}$$

For example, if $G = \Pi\{G_i : i \in I\}$ is a basic open subset of ΠX_i in the Tychonoff-product topology, then $|R(G)| < \aleph_0$.

B. Definition. Let $X = \Pi\{X_i : i \in I\}$. The topology generated by the subsets of the form $W = \Pi\{W_i : i \in I\}$ where each W_i is open in X_i and $|R(W)| < k$ is called the k-box topology on the product X and it is denoted by $(X)_k$ where $k \geq \aleph_0$.

In particular, if $k = \aleph_0$, then the k-box topology on X is the Tychonoff product topology. If $k = |I|^+$, then the k-box topology on X is simply referred to as the box topology on X.

In the case of $|I| < \aleph_0$, the Tychonoff product topology and the box-topology are the same. Thus when we deal with k-box topologies, we consider only infinite products.

C. Notation. Let $X = \Pi\{X_i : i \in I\}$; then $X_{I'}$ denotes the sub-product $\Pi\{X_i : i \in I'\}$ where $I' \subseteq I$ and $\pi_{I'}$ denotes the projection map from

X to $X_{I'}$. In particular if $I' = \{i\}$ then we get the usual projection map $\pi_i : X \to X_i$.

D. <u>Remark</u>. Let $W = \Pi\{W_i : i \in I\}$ where W_i is a subset of X_i for each $i \in I$. Then we have the following:

(i) $|R(\pi_{I'}(W))| \leq |R(W)|$

(ii) $|R(\pi_{I'}^{-1}(W_{I'}))| = |R(W_{I'})|$

(iii) If $V \supset W$, then $\pi_i(V) = X_i$ for all $i \in I - R(W)$.

E. <u>Proposition</u>. Let $U = \Pi\{U_i : i \in I\}$ and let $V = \Pi\{V_i : i \in I\}$ where U_i and V_i are subsets of X_i for each $i \in I$. Then the following are equivalent:

(i) $U \cap V = \Phi$,

(ii) $U_i \cap V_i = \Phi$ for some $i \in R(U) \cap R(V)$,

(iii) $R(U) \cap R(V) \neq \Phi$ and $\pi_{I'}(U) \cap \pi_{I'}(V) = \Phi$ where $I \supseteq I' \supseteq R(U) \cap R(V)$.

<u>Proof</u>. (i) $\Rightarrow$ (ii) : Trivial,

(ii) $\Rightarrow$ (iii) : Trivial,

(iii) $\Rightarrow$ (i) : Let $I' = R(U) \cap R(V)$, then $U_i \cap V_i = \Phi$ for some $i \in R(U) \cap R(V)$ and hence $U \cap V = \Phi$.

F. <u>Proposition</u>. The projection map $\pi_{I'} : (X)_k \to (X_{I'})_k$ has the following properties:

(i) $\pi_{I'}$ is onto (assuming the axiom of choice),

(ii) $\pi_{I'}$ is continuous,

(iii) $\pi_{I'}$ is open.

Proof. The above properties follow from remark C.

Furthermore we note the following: Let $X(I') = \{x \in X : x_i = a_i$ for $i \in I - I'\}$ where $a = (a_i)$ is a fixed point in X. Then $X(I')$ is homeomorphic to $(X_{I'})_k$, as a subspace of $(X)_k$. This follows from the fact that $\pi_{I'}/X(I')$ is a homeomorphism from $X(I')$ to $(X_{I'})_k$.

G. Definition. Let $X = \Pi\{X_i : i \in I\}$ and let $a = (a_i)$ be a fixed point in X. Then we define the γ-weak sum of $\{X_i : i \in I\}$ as follows:

$$\gamma(X) = \{x \in X : |\{i \in I : x_i \neq a_i\}| < \gamma\}$$

where γ is an infinite cardinal. We note that $\gamma(X) = \cup\{X(I') : I' \in I^{\underline{\gamma}}\}$ where $I^{\underline{\gamma}} = \{I' : I' \subseteq I$ and $|I'| < \gamma\}$. The γ-weak sum $\gamma(X)$ depends on the point $a = (a_i)$ and in the text we consider $\gamma(X)$ with respect to a fixed $a = (a_i)$, unless otherwise stated.

H. Theorem. Let $X = \Pi\{X_i : i \in I\}$. Then $\gamma(X)$ is a dense subspace of $(X)_k$ where $k \leq \gamma$.

Proof. Let $W = \Pi\{W_i : i \in I\}$ be a basic open set in $(X)_k$. Let $p = (p_i)$ where

$$p_i = \begin{cases} a_i, & i \in I - R(W) \\ w_i \in W_i, & i \in R(W). \end{cases}$$

Then $p \in W$ and we shall show that $p \in \gamma(X)$. Consider

$$|\{i \in I : p_i \neq a_i\}| \leq |R(W)| < k \leq \gamma$$

and hence $p \in \gamma(X)$. Therefore $\gamma(X)$ is a dense subspace of $(X)_k$.

As a special case if we take $k = \gamma = \aleph_0$, then the weak-topological sum, $\aleph_0(\Pi X_i)$ is a dense subspace of X in the product topology.

I. Example. Let $X_i = \mathbb{N}$, for $i = 1,2,\ldots,$ then $\aleph_0(\Pi X_i)$ is not dense in $(\Pi X_i)_{\aleph_1}$ because if $a_i \neq p_i$ for all $i \in I$, then $p \notin \aleph_0(\Pi X_i)$ but $\{p\}$ is an open subset of $(\Pi X_i)_{\aleph_1}$. Thus, the condition $k \leq \gamma$ cannot be relaxed in G.

CHAPTER II: m - n COMPACT SPACES

1. Some Properties of m-n Compact Spaces

1.1. Basic Properties

Compactness, countable compactness and the Lindelöf property are special cases of a more general concept: m-n compactness. In this section we shall study some properties of m-n compact spaces. Our notation follows that of Noble [53].

A. Definition. Let m and n be infinite cardinals with $m \geq n$. A topological space X is said to be m-n compact if and only if every open cover $\mathcal{U}$ of X of cardinality $\leq m$ has a subcover of cardinality $< n$.

We say X is ∞-n compact if and only if X is m-n compact for all $m \geq n$.

B. Special Cases.

(i) $\infty - \aleph_0$ compact spaces $\equiv$ compact spaces

(ii) $\aleph_0 - \aleph_0$ compact spaces $\equiv$ countably compact spaces

(iii) $\infty - \aleph_1$ compact spaces $\equiv$ Lindelöf spaces.

We will show next that m-n filters can be used to characterize m-n compact spaces. First we require a definition.

C. <u>Definition</u>. Let $\mathcal{F}$ be a filter on a topological space X. Then we define the <u>adherent</u> of $\mathcal{F}$ as

$$\text{ad } \mathcal{F} = \cap\{\bar{F} \mid F \in \mathcal{F}\} .$$

The following lemma is proved by Gal [18], but the proof will be reproduced here since our terminology is different.

D. <u>Lemma</u>. Let X be a topological space. Then the following are equivalent:

(i) X is m-n compact,

(ii) every family of closed subsets of X with the $<n$-intersection property also has the $\leq m$-intersection property,

(iii) for every m-n filter $\mathcal{F}$ on X, ad $\mathcal{F} \neq \Phi$.

<u>Proof</u>. (i) $\Rightarrow$ (ii). Let $\{F_i : i \in I\}$ be a family of closed subsets of X with the $<n$-intersection property. Then $\{X - F_i : i \in I\}$ does not contain an open cover of X of size $\leq m$ and hence $\{F_i : i \in I\}$ has the $\leq m$-intersection property.

(ii) $\Rightarrow$ (iii). Let $\mathcal{F}$ be an m-n filter on X and let $\mathcal{F}_B$ be a base of $\mathcal{F}$ with $|\mathcal{F}_B| \leq m$. Then $\bar{\mathcal{F}}_B = \{\bar{F} : F \in \mathcal{F}_B\}$ is a collection of closed subsets of X with the $<n$-intersection property and hence ad $\mathcal{F} = \cap\, \bar{\mathcal{F}}_B \neq \Phi$.

(iii) $\Rightarrow$ (i) Suppose X is not m-n compact. Then, there exists an open cover $\{U_i : i \in I\}$ of X with $|I| \leq m$ and no subcover of cardinality less than n. Hence,

$$\mathcal{F}_B = \{X - (\cup\{U_i : i \in J \in P_{<\aleph_0}(I)\})\}$$

is an m-n filter base on X but ad $F_B = \Phi$. Hence, we have a contradiction.

E. Corollary. If n is regular and $m^{\bar{n}} = m$, then a topological space X is m-n compact if and only if every m-n stable filter on X has a non-empty adherent.

F. Definition. A topological space is said to be $<n$-discrete if and and only if every point of X has a $<n$-stable neighbourhood base.

G. Theorem. Let X be a T_3, $<n$-discrete space. Suppose each $x \in X$ has a m-n compact neighbourhood. If n is regular, $m^{\bar{n}} = m$ and $\psi(X) \leq m$, then $\chi(X) \leq m$.

Proof. Let $\mathcal{B}_x$ be a collection of open neighbourhoods of $x \in X$ with $|\mathcal{B}_x| \leq m$ and $\{x\} = \cap\{\bar{B} : B \in \mathcal{B}_x\}$. Let K be an m-n compact neighbourhood of x. Let $\mathcal{V}_x = \{U : U = \cap\mathcal{U}, \mathcal{U} \in P_{<n}(\mathcal{B}_x)\}$. Then we shall show that $\mathcal{V}_x|K = \{U \cap K : U \in \mathcal{V}_x\}$ is a neighbourhood base at x. Let V be any open neighbourhood of x and suppose $(X-V) \cap (U \cap K) \neq \Phi$ for every $U \in \mathcal{V}_x$. Then $\mathcal{V}_x|(K \cap X-V)$ is a m-n stable filter base on K and

$$\begin{aligned} ad_K(\mathcal{V}_x|(K \cap X-V)) &\subseteq (\cap \bar{\mathcal{V}}_x) \cap (X - V) \cap K \\ &\subseteq (\cap \bar{\mathcal{B}}_x) \cap (X - V) \cap K \\ &= \{x\} \cap (X - V) = \Phi. \end{aligned}$$

By corollary E, this is a contradiction. Now it follows that $\chi(X) \leq m$.

G. Corollary. Let X be a T_3, m-n compact, $<$n-discrete space. If n is regular, $m^{\bar{n}} = m$ and $\psi(X) \leq m$, then $\chi(X) \leq m$.

1.2. Projection Maps

We shall show that, under suitable conditions on factor spaces, the projection map parallel to an m-n compact factor is a closed map.

A. Proposition. Let the projection $\pi_X : X \times Y \to X$ be a closed map. Suppose Y is a $<$n-discrete, T_1-space and $\chi(p,X) = n$ $(\geq \aleph_0)$ for some $p \in X$. Then Y is n-n compact.

Proof. Suppose Y is not n-n compact. Then there exists a n-n filter base $\mathcal{F}$ on Y such that $\cap \bar{\mathcal{F}} = \Phi$. Let

$$\mathcal{F} = \{F_\alpha : \alpha \in W(n)\},$$

$$K_\alpha = \cap \{F_\beta : \beta \leq \alpha\}$$

where $W(n)$ is the least ordinal of cardinality n. Let $\mathcal{V}_p = \{V_\alpha : \alpha \in W(n)\}$ be an open neighbourhood base at p and let $S_\alpha = X - V_\alpha$. Then,

$$p \in \overline{\cup\{S_\alpha : \alpha \in W(n)\}}$$

and

$$p \notin \overline{\cup\{S_\alpha : \alpha < \alpha_0\}}$$

for all $\alpha_0 \in W(n)$. Let $y \in Y$. Since $\cap \bar{F} = \Phi$, there exists an $\alpha_0 \in W(n)$ and an open set W of Y such that $y \in W$ and $W \cap K_{\alpha_0} = \Phi$. We set,

$$F = \overline{\cup\{S_\alpha \times K : \alpha \in W(n)\}}$$

and note that $(p,y) \notin F$, and hence $p \notin \pi_X(F)$. But

$$\overline{\pi_X(F)} \supseteq \overline{\cup\{S_\alpha : \alpha \in W(n)\}}$$

and thus $p \in \overline{\pi_X(F)}$. This is a contradication because π_X is a closed map.

B. <u>Proposition</u>. If X is $<$ n-discrete and has character $\leq m$ and Y is m-n compact, then the projection $\pi_X : X \times Y \to X$ is a closed map.

<u>Proof</u>. Let $F \subset X \times Y$ be a closed set and $x \in \pi_X(F)$. Let $\mathcal{V}_x$ be a neighbourhood system at x in X with $|\mathcal{V}_x| \leq m$. We set

$$\mathcal{G} = \{\pi_X^{-1}(V) \cap F : V \in \mathcal{V}_x\}$$

and let $\pi_Y(\mathcal{G}) = \{\pi_Y(G) : G \in \mathcal{G}\}$. Then easily $\pi_Y(\mathcal{G})$ is a base for an m-n filter on Y and hence, by 2-<u>B</u>, we can find some $y \in \cap\{\overline{\pi_Y(G)} : G \in \mathcal{G}\}$. Now, if V and W are neighbourhoods of x and y, then W meets $\pi_Y(\pi_X^{-1}(V) \cap F)$ and hence $\pi_Y^{-1}(W) \cap \pi_X^{-1}(V) \cap F \neq \Phi$. Thus every basic neighbourhood $V \times W$ of (x,y) in $X \times Y$ meets F and hence $(x,y) \in \bar{F} = F$. Then clearly, $x \in \pi_X(F)$.

2. Products of Lindelöf Spaces

Our main aim is to prove that a product of Lindelöf, T_3, P-spaces is Lindelöf whenever it is quasi paracompact.

2.1. Basic Terminology

We shall define quasi paracompactness and show that the class of quasi paracompact spaces is larger than the class of paracompact, T_2-spaces.

A. Definition. Let X be a topological space. A collection $\mathcal{C}$ of subsets of X is said to be $<n$-closure preserving if and only if for every $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| < n$,

$$\overline{\cup\{C : C \in \mathcal{C}'\}} = \cup\{\bar{C} : C \in \mathcal{C}'\}$$

where $n \geq \aleph_1$.

B. Examples.

(i) Every locally finite collection is $<n$-closure preserving for every n.

(ii) Let X be a $<n$-discrete space. Then any collection of subsets of X is $<n$-closure preserving.

C. Definition. A T_2-space X is said to be quasi n-paracompact if and only if every open cover of X has an open, $<n$-closure preserving refinement.

A T_2-space X is said to be quasi paracompact if and only if X is quasi $\aleph_1$-paracompact.

The class of quasi n-paracompact spaces contains the class of paracompact spaces and the class of < n-discrete, T_2-spaces. Thus the class of quasi paracompact spaces is larger than the class of paracompact, T_2-spaces.

2.2. Countable Products

We shall consider the productivity of the Lindelöf property when the product is normal and T_1.

A. Theorem (J.E. Vaughan [73]). If each X_i is m-n compact, <n-discrete and has character $\leq m$ for $i = 1,2,...$ and if n is regular and $m^n = m$, then $\Pi\{X_i : i = 1,2,...\}$ is m-n compact.

B. Corollary. Let n be a regular cardinal. If each X_i is ∞-n compact and < n-discrete for $i = 1,2,...,$ then $\Pi\{X_i : i = 1,2,...\}$ is ∞-n compact.

Since $\aleph_1$ is a regular cardinal the following is a special case of B:

C. Theorem (N. Noble [53]). A countable product of Lindelöf P-spaces if Lindelöf.

Next, we shall consider arbitrary products which satisfy additional conditions. In this direction the most remarkable theorem has been proved by A.H. Stone. The reference is [66] and we shall quote the theorem.

D. Theorem. If $X = \Pi\{X_i : i \in I\}$ is a normal space and if each X_i is T_1, then all but countably many of the X_i are countably compact.

E. Proposition.

(i) Let X be a T_1-space. If X is a countably compact, P-space, then X is finite.

(ii) If X is compact and Y is ∞-n compact, then $X \times Y$ is ∞-n compact.

Proof. (i) If X has a countably infinite subset B, then B is closed and discrete. This is a contradiction because X is countably compact. Hence X is finite.

(ii) Standard methods work.

We shall show that normality and paracompactness coincide in the product space, when the factor spaces are Lindelöf, T_3 P-spaces

F. Proposition. If X is any product of Lindelöf, T_3, P-spaces, then the following are equivalent:

(i) X is Lindelöf and T_3,

(ii) X is normal,

(iii) all but countably many factor spaces are countably compact.

Proof. (i) $\Rightarrow$ (ii): Follows from a standard result.

(ii) $\Rightarrow$ (iii): Follows from theorem D.

(iii) $\Rightarrow$ (i): Since all spaces are compact except for countably many, the result follows from theorem C and E-(ii).

G. Proposition. Let $X = \Pi\{X_i : i \in I\}$ be a normal T_1-space. If each X_i is ∞-n compact and $<$n-discrete, where n is a regular cardinal then X is ∞-n compact.

Proof. By theorem D, there exists a countable subset J of I such that $X = X_J \times X_{I-J}$ where X_J is a countable product of ∞-n compact, $<$n-discrete spaces and X_{I-J} is an arbitrary product of finite spaces. Hence, by E-(ii), X is ∞-n compact.

2.3. Large Products

We shall prove that the result 2-G is valid when the product is quasi n-paracompact.

A. Lemma. Let X be a quasi n-paracompact, regular space. If X has a dense, ∞-n compact subspace, then X is ∞-n compact.

Proof. Let C be a dense ∞-n compact subspace of X. Let $\mathcal{U}$ be an open cover of X. Since X is regular, we can find a refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that $\bar{V} \subset U$. Now, since X is quasi n-paracompact, $\mathcal{V}$ has an open $<$n-closure preserving refinement $\mathcal{W}$. Since $C \subseteq X \subseteq \cup\,\mathcal{W}$ and C is ∞-n compact, there exists a $\mathcal{W}' \subseteq \mathcal{W}$ such that $|\mathcal{W}'| < n$ and $C \subseteq \cup\mathcal{W}'$. Thus $X = \overline{\cup\mathcal{W}'}$. Let $\mathcal{W}' = \{W_i : i \in I\}$ where $|I| < n$. Then,

$$X = \cup\{\bar{W}_i : W_i \in \mathcal{W}' , i \in I\}$$

$$= \cup\{\bar{V}_i : V_i \in \mathcal{V} , i \in I\}$$

$$= \cup\{U_i : U_i \in \mathcal{U} , i \in I\}$$

Since $|I| < n$, X is ∞-n compact.

B. Corollary. Every paracompact T_2-space with a dense ∞-n compact subspace is ∞-n compact.

C. Corollary. Every separable, paracompact T_2-space is Lindelöf.

D. Theorem [theorem 2.3 of [13]). Let $m \geq n > \gamma \geq \aleph_0$ and $n \geq \alpha \geq \aleph_0$ with n regular and strongly γ-inaccessible. Let $\{X_i : i \in I\}$ be a family of spaces such that $(\Pi\{X_i : i \in I'\})_\alpha$ is m-n compact for all $I' \in P_\alpha(I)$. Then $(\gamma(\Pi\{X_i : i \in I\}))_\alpha$ is m-n compact with respect to the canonical basis.

E. Corollary. Let $n > \gamma \geq \aleph_0$ with n regular and strongly γ-inaccessible. Let $\{X_i : i \in I\}$ be a family of spaces such that $\Pi\{X_i : i \in I'\}$ is ∞-n compact for all $I' \in P_\gamma(I)$. Then $\gamma(\Pi\{X_i : i \in I\})$ is ∞-n compact.

F. Proposition. Let $\{X_i : i \in I\}$ be a family of spaces such that each X_i is ∞-n compact and $<n$-discrete, where n is a regular cardinal. Then $\aleph_0(\Pi\{X_i : i \in I\})$ is ∞-n compact.

Proof. We note that, by 2-B, every finite sub-product is ∞-n compact and hence we can apply 3-E, taking $\gamma = \aleph_0$.

G. Theorem. Let $X = \Pi\{X_i : i \in I\}$ be a quasi n-paracompact space. Let n be a regular cardinal and suppose each X_i is ∞-n compact,

$<n$-discrete and T_3; then X is ∞-n compact.

Proof. Since $\aleph_0(\Pi\{X_i : i \in I\})$ is a dense, ∞-n compact subspace of X, the result follows by lemma 3-A.

H. Corollary. Let $X = \Pi\{X_i : i \in I\}$ be a quasi paracompact space. If each X_i is a Lindelöf, T_3 P-space, then X is a Lindelöf space.

3. Locally ∞-n Compact Spaces

We define locally ∞-n compact spaces and study the basic properties of this class.

A. Definition. A topological space X is said to be locally ∞-n compact if every $x \in X$ has a neighbourhood base consisting of ∞-n compact subsets, where $n \geq \aleph_0$.

A locally $\infty - \aleph_0$ compact space is called a locally compact space and a locally $\infty - \aleph_1$ compact space is called a locally Lindelöf space.

B. Examples.

(i) The set of rationals, as a subspace of $\mathbb{R}$, is locally Lindelöf but not locally compact.

(ii) Let X be an uncountable discrete space. We adjoin an extra point p to X and specify its neighbourhoods to be $A \cup \{p\}$ where $A \subset X$ and $X - A$ is countable. The neighbourhoods of points of X remain the same. The space $X^* = X \cup \{p\}$ is called the one-point Lindelöf extension of X.

We note the following properties:

(a) X^* is Lindelöf (locally Lindelöf),

(b) X^* is T_3,

(c) X^* is a P-space,

(d) If $A \subset X^*$ and A is compact, then $|A| < \aleph_0$.

By property (d) it is clear that X^* is not locally compact.

C. Some Properties. The following properties are useful and they are easy consequences of the definition A:

(i) In the presence of regularity, to show that a space X is locally ∞-n compact it is sufficient to find one ∞-n compact neighbourhood at each $x \in X$. Thus every ∞-n compact, T_3 space is locally ∞-n compact.

(ii) Every open subspace of a regular, locally ∞-n compact space is locally ∞-n compact.

(iii) Let X be a $<n$-discrete T_2-space. Then every locally ∞-n compact subset is the intersection of an open set and a closed set.

(iv) Let X be a locally ∞-n compact, T_3, $<n$-discrete space. If A is dense in X, then A is locally ∞-n compact if and only if A is open.

D. Definition. A topological space X is said to be a k(n)-space if for each $A \subset X$, the set A is open in X if and only if $A \cap Z$ is open in Z for every ∞-n compact subset Z of X.

A $k(\aleph_0)$-space is called a k-space and a $k(\aleph_1)$-space is called an L-space.

E. Proposition. Every locally ∞-n compact space is a $k(n)$-space.

Proof. Suppose X is a locally ∞-n compact space and $A \cap Z$ is open in Z for every ∞-n compact subset Z of X. Let $a \in A$ and let M be a ∞-n compact neighbourhood of a. Then we note that $A \cap \operatorname{Int} M = (A \cap M) \cap \operatorname{Int} M$ is open in X, and contains a. Since

discrete spaces are locally ∞-n compact for any n, the property locally ∞-n compact need not be preserved by continuous maps.

F. Note. Continuous open maps preserve local ∞-n compactness but not continuous closed maps.

G. Theorem. Let X be a T_3-space. Then X is a k(n)-space if and only if X is a quotient of some locally ∞-n compact space.

Proof. Suppose X is a k(n)-space. Let $\mathcal{B}(n)$ denote the collection of all ∞-n compact subspaces of X. Let $|\mathcal{B}(n)| = k$. For each $B \in \mathcal{B}(n)$ we set

$$B(t) = B \times t \quad \text{where} \quad t \in k.$$

Then Y, the topological union of $\{B(t) : t \in k\}$, is locally ∞-n compact and the map $f : Y \to X$ defined by $f(x,t) = x$ is a quotient map. This completes the proof of necessity.

Suppose $p : Y \to X$ is a quotient map and Y is locally ∞-n compact. Let U be a subset of X such that $U \cap B$ is open in B for every $B \in \mathcal{B}(n)$. Let $y \in p^{-1}(U)$ and let V be a locally ∞-n compact neighbourhood of y. Then $U \cap p(V)$ is open in $p(V)$ and hence there exists an open subset G of X such that $U \cap p(V) = p(V) \cap G$. Now we note that

$$y \in p^{-1}(G) \cap \operatorname{Int} V \subseteq p^{-1}(U).$$

Hence $p^{-1}(U)$ is open in Y. Since p is a quotient map, U is open in X. This proves the sufficiency.

4. Tightness in Product Spaces

We consider the following question:

Is $\partial(X \times Y) \leq \partial(X) \cdot \partial(Y)$ where

X and Y are any two topological spaces?

4.1. Finite Products

We shall show that for certain classes of spaces, $\partial(X \times Y) = \partial(X) \cdot \partial(Y)$.

A. Definition. Let $p \in X$, $S \in X$ and $p \in \bar{S}$. Then, we define

$$\partial(p,S,X) = \min\{|M| : M \subset S \text{ and } p \in \bar{M}\}$$

$$\partial(p,X) = \sup\{\partial(p,S,X) : p \in \bar{S}\}$$

and $$\partial(X) = \sup\{\partial(p,X) : p \in X\} + \aleph_0$$

which is called the tightness of X.

B. Some Terminology. If $R = \{R_i : i \in I\}$ is any collection of topological spaces, we define,

$$\partial_I(R) = \sup\{\partial(R_i) : i \in I\}$$

We say, the product $R = \Pi\{R_i : i \in I\}$ preserve tightness if and only if

$$\partial(\Pi R_i) \leq |I| \cdot \partial_I(R).$$

For finitely many $X_1, X_2, \ldots X_n$, this reduces to,

$$\partial(\Pi X_i) \leq \partial(X_1) \cdot \partial(X_2) \ldots \partial(X_n)$$

C. Definition. A space X is called a GC(n,i)-space for $i = 1,2,3,4$ if X is an ∞-n compact, $<$ n-discrete, T_i-space.

A space X is called a GLC(n,i)-space for $i = 1,2,3,4$ if X is a locally ∞-n compact, $<$ n-discrete, T_i-space. Always, $GC(n,i) \subseteq GLC(n,i)$ for $i = 1,2,3,4$.

D. Proposition. If X is a GC(n,3) space, then X is (strongly) paracompact and T_2.

Proof. The case $n = \aleph_0$ is straight-forward. We shall thus consider only $n \geq \aleph_1$. Since X is a T_3 P-space, easily X is zero-dimensional. Thus, if $\mathcal{U}$ is an open cover of X, then we can find a refinement $\mathcal{V}$ of $\mathcal{U}$, consisting of $c\ell$-open sets with $|\mathcal{V}| < n$. Let $\mathcal{V} = \{V_\alpha : \alpha \subset k \in W(n)\}$. Now we set $W_\alpha = \mathcal{V}_\alpha - \cup\{V_\beta : \beta < \alpha\}$ for each $\alpha \in k$. Then, clearly $\{W_\alpha : \alpha \in k\}$ is an open star-finite refinement of $\mathcal{U}$. Hence X is (strongly) paracompact and T_2.

Next, we prove a basic result which generalizes Juhász's lemma for the compact case (page 113, [34]).

E. Lemma. If X is a $<$ n-discrete T_1-space and Y is an ∞-n compact regular space, then

$$\partial(X \times Y) \leq \partial(X) \cdot \partial(Y)$$

Proof: Let $k = \partial(X) \cdot \partial(Y)$ and suppose $H \subset X \times Y$ is k-closed. It suffices to show H is closed.

Let $(p,q) \in \bar{H}$. If $T = H \cap (\{p\} \times Y)$, then T is k-closed and hence closed in $\{p\} \times Y$, and hence closed in $X \times Y$. We need only show $q \in \pi_Y(T)$.

Suppose $q \notin \pi_Y(T)$. Since $\pi_Y \mid \{p\} \times Y$ is a homeomorphism, $\pi_Y(T)$ is closed in Y. Let V be a closed neighbourhood of q such that $V \cap \pi_Y(T) = \phi$. Note that since $X \times V$ is a neighbourhood of (p,q) and $(p,q) \in \bar{H}$, we have $(p,q) \in \overline{(X \times V) \cap H}$. But $(X \times Y) \cap H$ is a closed subset of H, and hence k-closed in $X \times Y$. Since Y is ∞-n compact and X is $<$n-discrete, π_X is closed by 1.2 B. Thus $\pi_X[(X \times V) \cap H]$ is k-closed and hence closed in X. Then, since π_X is continuous,

$$p \in \pi_X\overline{[(X \times V) \cap H]} \subset \overline{\pi_X[(X \times V) \cap H]}$$
$$= \pi_X[(X \times V) \cap H] .$$

So for some $v \in V$, $(p,v) \in H$. But then $v \in \pi_Y(T) \cap V$, a contradiction.

F. <u>Theorem.</u> If X is $<$n-discrete, T_1 and Y is locally ∞-n compact, T_3, then

$$\partial(X \times Y) \leq \partial(X) \cdot \partial(Y).$$

<u>Proof</u>. Let $k = \partial(X) \cdot \partial(Y)$ and let $H \subset X \times Y$ be k-closed. Choose $(p,q) \in \bar{H}$. Let V be any closed ∞-n compact neighbourhood of q. Then $X \times V$ is closed in $X \times Y$ and $\partial(X \times V) \leq k$ by D. But $H \cap (X \times V)$ is k-closed in $X \times V$, and thus closed in $X \times V$, and $(p,q) \in Cl_{X \times V} H \cap (X \times V) = H \cap (X \times V)$. Thus $(p,q) \in H$.

G. Corollary. If $X_1, X_2, \ldots, X_n$ are GLC(n,3) spaces, then

$$\partial\left(\prod_{i=1}^{n} X_i\right) \leq \partial(X_1) \cdot \ldots \cdot \partial(X_n)$$

Proof. By induction, noting that

$$X_1 \times \ldots \times X_k = (X_1 \times \ldots \times X_{k-1}) \times X_k$$

and that $X_1 \times \ldots \times X_{k-1}$ is $< n$ discrete and T_1, while X_k is locally ∞-n compact and T_3, so that

$$\partial(X_1 \times \ldots \times X_k) \leq \partial(X_1 \times \ldots \times X_{k-1}) \cdot \partial(X_k)$$

by E.

Thus, finite products of locally compact, T_2-spaces (= GLC($\aleph_0$,3)) preserve tightness and so also do finite products of locally Lindelöf T_3 P-spaces (= GLC($\aleph_1$,3)), and so on. We shall consider infinite products in the next section.

4.2. Large Products

In this section, we shall extend the results of the previous sections to arbitrary products.

A. Notation. Let $R = \Pi\{R_i : i \in I\}$ and suppose $J \subset I$. The subproduct $\Pi\{R_i : i \in J\}$ of R will be denoted by R_J and the projection of R onto R_J will be denoted by π_J. For $a \in R$ and $A \subset R$, the images $\pi_J(a)$ and $\pi_J(A)$ will be denoted by a_J and A_J respectively.

B. Proposition. If each finite subproduct of a product $R = \Pi\{R_i : i \in I\}$ preserves tightness, then R preserves tightness.

Proof. Let $R = \{R_i : i \in I\}$ and set $k = |I|\partial_I(R)$. Let $I = \{J \subset I : |J| < \aleph_0\}$. Suppose $A \subset R$ is k-closed and $a \in \bar{A}$. Then for $J \in I^{(F)}$, a_J belongs to $\pi_J(\bar{A}) \subset \bar{A}_J$. Since $\partial(R_J) \leq K$, we can find $B_J \subset A_J$ with $|B_J| \leq k$ such that $a_J \in \bar{B}_J$.

For each $b \in B_J$, choose $x_b \in A$ so that $\pi_J(x_b) = b$, and set

$$C_J = \{x_b \mid b \in B_J\}.$$

Clearly $|C_J| = |B_J| \leq k$ and hence if $C = \cup\{C_J \mid J \in I^{(F)}\}$, then $C \subset A$ and $|C| \leq k \cdot |I| = k$.

But $a \in \bar{C}$. For if $U = U_J \times \prod_{i \notin J} R_i$ is a basic open neighbourhood of a, then since $\pi_J(a) \in \overline{B_J} = \overline{\pi_J(C_J)}$, we have $U_J \cap \pi_J(C_J) \neq \phi$, and thus $U \cap C_J \neq \phi$, whence $U \cap C \neq \phi$.

Thus $a \in A$, and A is closed.

C. Theorem. If R_i is a GLC(n,3) space for each $i \in I$, then $\partial(\Pi R_i) = |I|\partial_I(R)$.

Proof. Apply B and 3-F.

In 5.9 of [34], it is shown that for any collection of compact, T_2-spaces $\{R_i : i \in I\}$, $\partial(\Pi R_i) = |I| \cdot \partial_I(R)$. The special cases of C, generalize this result.

D. Corollary. If R_i is locally compact and T_2 for each $i \in I$, then $\partial(\Pi R_i) = |I| \cdot \partial_I(R)$.

Proof. Locally compact, $T_2 \equiv GLC(\aleph_0, 3)$.

E. Corollary. If R_i is a locally Lindelöf, T_3, P-space for each $i \in I$, then $\partial(\Pi R_i) = |I| \cdot \partial_I(R)$.

Proof. Locally Lindelöf, T_3, P-space $\equiv GLC(\aleph_1, 3)$.

CHAPTER III: WEAKLY m-n COMPACT SPACES

1. Some Properties of Weakly m-n Compact Spaces

1.1. Basic Facts

The properties H(i), feebly compact and weakly Lindelöf are special cases of the general concept weak m-n compactness. In this section we shall study basic properties of weakly m-n compact spaces.

A. Definition. A topological space X is said to be weakly m-n compact if and only if every open cover of X of cardinality $\leq m$ has a sub-family of cardinality $< n$ with dense union, where $m \geq n \geq \aleph_0$.

A topological space X is said to be weakly ∞-n compact if and only if X is weakly m-n compact for each $m \geq n$.

B. Special Cases (see [62] and [69]).

(i) Weakly ∞- $\aleph_0$ compact spaces $\equiv$ H(i) - spaces

(ii) Weakly $\aleph_0$ - $\aleph_0$ compact spaces $\equiv$ feebly compact spaces

(iii) Weakly ∞- $\aleph_1$ compact spaces $\equiv$ weakly Lindelöf spaces.

A T_2, H(i)-space is called an H-closed space.

C. Definition. A subset E of a topological space X is said to be weakly m-n compact if it is weakly m-n compact in its subspace topology.

A subset E of a topological space X is said to be relatively weakly m-n compact if and only if every X-open cover $\mathcal{U}$ of E with

$|\mathcal{U}| \leq m$, has a sub-family $\mathcal{V}$ with $|\mathcal{V}| < n$ and $E \subseteq \overline{\cup\mathcal{V}}$.

It is clear that every weakly m-n compact set is relatively weakly m-n compact. In general, the converse is not true, except for open subsets.

D. Example. Let X be a discrete subspace of $\beta\mathbb{N} - \mathbb{N}$, with cardinality c. Let $Y = X \cup \mathbb{N}$ as a subspace of $\beta\mathbb{N}$. Then X is a relatively weakly Lindelöf subset of Y but not weakly Lindelöf. Here X is a closed subset of Y.

Next, we shall study basic properties of weakly m-n compact spaces.

E. Proposition.

(i) Let n be an infinite cardinal. If $\{X_i : i \in I\}$ is a collection of relatively weakly m-n compact subsets of X, then $\cup\{X_i : i \in I\}$ is relatively weakly m-n compact, provided $|I| < cf(n)$.

(ii) If X has a dense, relatively weakly m-n compact subset, then X is weakly m-n compact.

(iii) A continuous image of a weakly m-n compact space is weakly m-n compact.

(iv) A regular closed subspace of a weakly m-n compact space is weakly m-n compact.

Proof. (i), (ii) and (iii) are easy consequences of the definition A. We shall prove (iv).

Let B be a regular closed subset of a weakly m-n compact

space X. Let $\mathcal{V}$ be an open cover of B of cardinality $\leq m$. Then for each $V \in \mathcal{V}$ we select an open subset U_V of X such that $V = U_V \cap B$. Now $\mathcal{U} = \{U_V : V \in \mathcal{V}\} \cup \{X - B\}$ is an open cover of X and $|\mathcal{U}| \leq m$. Since X is weakly m-n compact, there exists a $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| < n$ and

$$X = \overline{\cup\{U_V : V \in \mathcal{V}'\} \cup \{X - B\}}\,.$$

It follows that

$$B = C\ell_X(\mathrm{Int}_X\, B) \subseteq \overline{\cup\{V : V \in \mathcal{V}'\}}\,.$$

Thus B is a weakly m-n compact subspace.

F. Proposition. Let $p : X \to Y$ be a function with the following properties:

(i) p is closed,

(ii) $p^{-1}(y)$ is compact for every $y \in Y$,

(iii) $p^{-1}(\bar{V}) \subseteq \overline{p^{-1}(V)}$ for every open subset V of Y.

Suppose Y is weakly m-n compact, then X is weakly m-n compact.

Proof. Let $\mathcal{V}$ be an open cover of X of cardinality $\leq m$. Let $\mathcal{W} = \{\cup\mathcal{U} : \mathcal{U} \in P_{<\aleph_0}(\mathcal{V})\}$. For each $W \in \mathcal{W}$, we set $G_W = Y - p(X-W)$. Note that $\{G_W : W \in \mathcal{W}\}$ is an open cover of Y of cardinality $\leq m$. Thus there exists a $\mathcal{W}' \subseteq \mathcal{W}$ such that $|\mathcal{W}'| < n$ and $Y = \overline{\cup\{G_W : W \in \mathcal{W}'\}}$. Now, since $p^{-1}(G_W) \subseteq W$, $X = \overline{\cup\{W : W \in \mathcal{W}'\}}$. Hence X is weakly m-n compact.

G. Corollary. If X is weakly m-n compact and Y is compact, then $X \times Y$ is weakly m-n compact.

Proof. The projection map $\pi_X : X \times Y \to X$ satisfies the properties (i), (ii) and (iii) of F.

1.2. Special Properties

In this section we shall study special properties of weakly m-n compact spaces.

A. Theorem. A topological space X is weakly m-n compact if every proper regular-closed subset of X is relatively m-n compact.

Proof. Let $\mathcal{U}$ be an open cover of X of cardinality $\leq m$. If $U \in \mathcal{U}$ is non-empty and $U \neq X$, take $B = c\ell_X(X-\bar{U})$. Then clearly B is a regular closed subset of X, and hence there exists a $\mathcal{U}' \subseteq \mathcal{U}$ of cardinality $< n$ such that,

$$B \subseteq \overline{\cup\{V : V \in \mathcal{U}'\}} .$$

Thus

$$X = \overline{\cup\{V : V \in \mathcal{U}'\}} \cup (\bar{U})$$

and since $|\mathcal{U}' \cup \{U\}| < n$, we are done.

Combining theorem A with the results of section 1, we have:

B. Corollary. A topological space X is weakly m-n compact if and only if every proper regular closed subspace is weakly m-n compact.

We shall show that the property weak m-n compactness is not closed hereditary. This is one of the intrinsic differences between weak m-n compactness and m-n compactness.

C. Example. Whenever $k > n > \aleph_0$, there exists a T_2 space V such that

(i) V is weakly ∞-n compact, and

(ii) V has a closed discrete subspace of cardinality k.

To see this, let X be any T_2 space with disjoint dense subsets A and B, where $|B| < n$, and let Y be the discrete space of cardinality k. We form V by adjoining to the product space $T = A \times Y$ a copy of B, with neighbourhoods of $b \in B$ in V taking the form

$$[U \cap B] \cup [(U \cap A) \times Y]$$

where U is a neighbourhood of b in X.

Note that, easily, $\{a\} \times Y$ is a closed discrete subspace of V of cardinality k, for any $a \in A$. To see that V is weakly ∞-n compact, let $\mathcal{W}$ be any open cover of V. For each $b \in B$, choose $W_b \in \mathcal{W}$ so that $b \in W_b$. Then $\{W_b \mid b \in B\}$ is an open set in V containing B, and it is easy to see that any open set in V containing B is dense in V. Thus, since $|\{W_b \mid b \in B\}| = |B| < n$, $\{W_b \mid b \in B\}$ is the required subcollection of $\mathcal{W}$.

D. Note. The above example is an abstraction of example 2.3 in [4].

E. Proposition. Let X be a regular, quasi n-paracompact space. If X is weakly ∞-n compact, then X is ∞-n compact.

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is regular, we can find an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for each $V \in \mathcal{V}$, $\bar{V} \subseteq U$ for some $U \in \mathcal{U}$. Now let $\mathcal{W}$ be an open, $<$n-closure preserving refinement of $\mathcal{V}$. Since X is weakly ∞-n compact, there exists a $\mathcal{W}' \subseteq \mathcal{W}$ of cardinality $< n$ and $X = \cup\{\bar{W} : W \in \mathcal{W}'\}$. For each $W \in \mathcal{W}$, $\bar{W} \subseteq U$ for some $U \in \mathcal{U}$. Thus, there exists $\mathcal{U}' \subset \mathcal{U}$ of cardinality $< n$ and $X = \cup\{U : U \in \mathcal{U}'\}$.

F. Corollary. The property weak ∞-n compactness coincides with ∞-n compactness in paracompact, T_2-spaces and in $<$n-discrete, regular spaces.

G. Example. Let X be the one-point Lindelöf extension of an uncountable discrete space of cardinality k. Let Y be the Alexandroff double of X. Then Y has a closed nowhere dense subspace E of cardinality k. Now, we define a new topology on Y in the following manner:

(i) the neighbourhoods of points of $Y-E$ remain unchanged,

(ii) the neighbourhoods of points of E take the form,

$$U_p^* = (U_p - E) \cup \{p\}$$

where U_p is a neighbourhood of p in the original topology of Y.

Calling this space Z, we note that, $C\ell_Z U_p^* = C\ell_Y U_p$ for all $p \in E$. Now, since Y is a Lindelöf, T_2, P-space, we have the following:

(i) Z is weakly Lindelöf,

(ii) Z is a P-space,

(iii) $L(Z) = c(Z) = k$,

(iv) Z is T_2.

Thus the class of weakly Lindelöf T_2, P-spaces is larger than the class of Lindelöf, T_2, P-spaces.

2. Some Characterizations

2.1. Filters

Be generalizing the intersection properties satisfied by open filters we wish to define strong m-n filters and using this generalized notion we shall characterize weakly m-n compact spaces (see [18]).

A. Definition. A collection $\mathcal{F}$ of subsets of a space X is said to have the $<$ n-strong intersection property if for each $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| < n$, we have $\mathrm{Int}(\cap \mathcal{F}') \neq \Phi$.

A collection $\mathcal{F}$ of non-empty subsets of a space X is said to be $<$ n-strongly stable if for each $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| < n$ we have $\mathrm{Int}(\cap \mathcal{F}') \supseteq F$ for some $F \in \mathcal{F}$.

B. Definition. A strong m-n filter $\mathcal{F}$ on a space X is a filter on X which has the $<$ n-strong intersection property and has a base $\mathcal{F}_B$ of cardinality less than or equal to m.

A strong m-n stable filter $\mathcal{F}$ on X is a filter on X which has the $<$n-strong stable property and has a base $\mathcal{F}_B$ of cardinality less than or equal to m.

C. Remark. Strong m-n filters are topological objects but m-n filters (filters) are set-theoretical objects; every strong m-n filter is an m-n filter but not conversely.

D. Example. Let X be a T_1-space with character $\leq$ k. Let x be a

non-isolated point of X. Then the neighbourhood system $\mathcal{V}_x$ at x is a m-n filter but not a strong m-n filter, where $m \geq n > k$.

E. Definition. A filter $\mathcal{F}$ is said to be of type k if $|F| \leq k$ for every $F \in \mathcal{F}_B$ where $\mathcal{F}_B$ is some filter base for $\mathcal{F}$.

F. Note. Every strong m-n stable filter on a k-separable space X induces a finer m-n stable filter of type k.

G. Lemma. Let $\mathcal{F}$ be a strong m-n filter on the space X. Let n be a regular cardinal number and let $m^{\underline{n}} = \sum\{m^k : k < n\} = m$. Then there exists a strong m-n stable filter $\mathcal{G}$ on X such that $\mathcal{G} \supseteq \mathcal{F}$.

Proof. Let $\mathcal{F}_B$ be a filter base for $\mathcal{F}$. Let $\mathcal{G}_B = \{\mathrm{Int}(\cap \mathcal{F}') : \mathcal{F}' \subset \mathcal{F}_B$ and $|\mathcal{F}'| < n\}$. Then $|\mathcal{G}_B| \leq |\mathcal{F}_B|^{\underline{n}} \leq m^{\underline{n}} = m$. Let $\mathcal{G}' \subset \mathcal{G}_B$ and $|\mathcal{G}'| < n$. Then $\mathrm{Int}(\cap \mathcal{G}') = \mathrm{Int}(\underset{<n}{\cap} (\mathrm{Int}(\underset{<n}{\cap} \mathcal{F}'))) \supseteq \mathrm{Int}(\underset{<n}{\cap} \mathcal{F}'')$ where $\mathcal{F}'' \subset \mathcal{F}_B$ and, since n is regular, $|\mathcal{F}''| < n$. Hence the filter $\mathcal{G}$ generated by $\mathcal{G}_B$ is a strong m-n stable filter on X and $\mathcal{G} \supseteq \mathcal{F}$.

The above lemma shows that every strong m-n filter can be embedded in a strong m-n stable filter provided n is regular and $m^{\underline{n}} = m$.

H. Theorem. Let X be a topological space. Then the following are equivalent:

(i) X is weakly m-n compact,

(ii) Every family of closed subsets of X with the < n-strong intersection property also has the $\leq m$-intersection property,

(iii) Every strong m-n filter on X has an adherent point.

Proof. (i) $\Rightarrow$ (ii): Let $\{F_i : i \in I\}$ be a family of closed subsets of X with the $< n$-strong intersection property. Then $\{X - F_i : i \in I\}$ contains no m-fold open cover of X and hence $\{F_i : i \in I\}$ has the $< m$-intersection property.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be a strong m-n filter on X. Let $\mathcal{F}_B$ be a base for $\mathcal{F}$ such that $|\mathcal{F}_B| \leq m$. Then $\{\bar{F} : F \in \mathcal{F}_B\}$ has the $< n$-strong intersection property and by the hypothesis, $\cap\mathcal{F}_B \neq \Phi$. Therefore $\mathcal{F}$ has an adherent point.

(iii) $\Rightarrow$ (i): If X is not weakly m-n compact, then there exists an m-fold open cover $\{G_i : i \in I\}$ of X with no dense sub-family of cardinality strictly less than n. Hence $\{X - G_i : i \in I\}$ has the $< n$-strong intersection property and therefore $\{X - G_i : i \in I\}$ is a filter sub-base for some strong m-n filter $\mathcal{F}$. But $\cap\bar{\mathcal{F}} = \cap\{X - G_i : i \in I\} = \Phi$. We have a contradiction and hence the result.

I. Corollary. Let n be a regular cardinal and let $m^{\bar{n}} = m$. Then a topological space X is weakly m-n compact if and only if every strong m-n stable filter on X has a non-empty adherent.

Proof. This follows from lemma G above.

2.2. Continuous Maps

We wish to characterize weakly m-n compact spaces using the local character of the points of $C\ell_Y f(X) - f(X)$ where $f : X \to Y$ is a continuous map. We will employ strong m-n stable filters.

A. Proposition. Let X be a $<n$-discrete, non weakly m-n compact T_2-space, where n is regular and $m^{\bar{n}} = m$. Then there exists a $<n$-discrete T_2-space Y and a continuous map $f : X \to Y$ such that

(i) $C\ell_Y\, f(X) = Y$ and

(ii) $Y - f(X)$ has a point of local character $\leq m$.

Proof. Let $F = \{F_k : k \in I\}$ be a strong m-n stable filter base on X such that $\cap \bar{F} = \Phi$. Let $Y = X \cup \{w\}$ where $w \notin X$, with the topology given by the open subsets of X together with the sets of the form $\text{Int}\, F_k \cup \{w\}$ where $k \in I$. Then the inclusion map $i : X \to Y$ is continuous and it is easy to see that Y satisfies the above properties.

B. Proposition. Let X be a weakly m-n compact space. Let Y be a $<n$-discrete T_2-space and let $f : X \to Y$ be a continuous map. Then $c\ell_Y\, f(X) - f(X)$ has no points of local character $\leq m$.

Proof. Suppose $C\ell_Y\, f(X) - f(X)$ has a point y of local character $\leq m$. Let ν be an open neighbourhood base at y with $|\nu| \leq m$. Then $\nu|f(X)$ is a strong m-n stable filter base on $f(X)$, but $\cap\{\bar{V} \mid V \in \nu\} = y \notin f(X)$. It follows that

$$\cap\{C\ell_{f(X)}(V \cap f(X)) \mid V \in \nu\} \subseteq \cap\{\bar{V} \mid V \in \nu\} \cap f(X)$$
$$= \phi.$$

But then $f(X)$ is not weakly m-n compact, a contradiction.

C. Theorem. A $<n$-discrete, T_2-space X is weakly m-n compact if and only if for each $<n$-discrete T_2-space Y and for each continuous

map $f : X \to Y$, $C\ell_Y f(X) - f(X)$ has no points of local character $\leq m$, where n is regular and $m^{\underline{n}} = m$.

Proof. Necessity follows from proposition B. To prove sufficiency, suppose X is not weakly m-n compact. Then by proposition A, we have a space Y which is $<n$-discrete and T_2 while $C\ell_Y f(X) - f(X)$ contains a point of local character $\leq m$. Hence we have a contradiction.

The following special case of C is of interest:

D. Corollary. A topological space X is feebly compact if and only if for each T_2-space Y and for each continuous map $f : X \to Y$, $C\ell_Y f(X) - f(X)$ is nowhere first countable in Y.

3. Products of Two Spaces

3.1. Machinery

We extend the method of Vaughan [73] to the setting of weakly m-n compact spaces.

A. Definition. A space X is said to satisfy the property $\bar{I}_{m,n}$ if and only if for every strong m-n filter base F on X, there exists a compact subset K of X and a strong m-n stable filter base G such that $G > F$ and $G > V_K$ where V_K is an open neighbourhood base of K.

B. Proposition.

(i) Let X be a space which satisfies the property $\bar{I}_{m,n}$. Then X is weakly m-n compact.

(ii) Let n be a regular cardinal and $m^{\bar{n}} = m$. Suppose X is weakly m-n compact, < n-discrete and has character $\leq m$. Then X satisfies the property $\bar{I}_{m,n}$.

Proof. (i) Let F be a strong m-n filter on X. Then since X satisfies the property $\bar{I}_{m,n}$, there exists a strong m-n stable filter G and X and a compact subset K of X such that $G > F$ and $G > V_K$. Hence $\cap\bar{F} \supseteq \cap\bar{G}$ and $(\cap\bar{G}) \cap K \neq \Phi$. Therefore $\cap F \neq \Phi$ and by 2-H, X is weakly m-n compact.

(ii) Let F be a strong m-n filter base on X. Then since n is regular and $m^{\bar{n}} = m$, there exists a strong m-n stable filter base F' on X such that $F' > F$. Since X is weakly m-n compact, there

exists some $x \in \cap \bar{F}'$; let V_x be an open neighbourhood base at x with $|V_x| \leq m$. Take $K = \{x\}$ and $G = F' \vee V_K$; then we note that G is a strong m-n stable filter base on X, $G > F' > F$ and $G > V_K$. Therefore X satisfies the property $\bar{1}_{m,n}$.

C. <u>Proposition</u>. Let n be regular and $m^{\bar{n}} = m$. Then every locally compact weakly m-n compact space X satisfies the property $\bar{1}_{m,n}$.

<u>Proof</u>. Let F be a strong m-n filter base on X. Since n is regular and $m^{\bar{n}} = m$, there exists a strong m-n stable filter base F' such that $F' > F$ and since X is weakly m-n compact, there exists a $x \in \cap F'$. Let K be a compact neighbourhood of x; then $F'|K = \{F \cap K : F \in F'\}$ is a strong m-n stable filter base on X and $F'|V > F' > F$ and $F'|K > V_K$. Hence X satisfies the property $\bar{1}_{m,n}$.

D. <u>Proposition</u>. Let $f : X \to Y$ be a bicontinuous (open and continuous) onto map. If X satisfies the property $\bar{1}_{m,n}$, then Y satisfies the property $\bar{1}_{m,n}$.

<u>Proof</u>. Let F be a strong m-n filter base on Y. Then since F is continuous, $f^{-1}(F) = \{f^{-1}(F) : F \in F\}$ is a strong m-n filter base on X and since X satisfies the property $\bar{1}_{m,n}$, there exists a strong m-n stable filter base G on X and a compact subset K of X such that $G > f^{-1}(F)$ and $G > V_K$. Then since f is open, $f(G)$ is a strong m-n stable filter base on Y and since f is onto $f(G) > F$ and $f(G) > V_{f(K)}$. Since f is continuous, $f(K)$ is a compact subset of Y and hence Y satisfies the property $\bar{1}_{m,n}$.

E. Corollary. Let $X = \Pi\{X_i : i \in I\}$ with the product topology. Suppose X satisfies the property $\bar{1}_{m,n}$; then every sub-product of X has the property $\bar{1}_{m,n}$.

Proof. Let $X_{I'} = \Pi\{X_i : i \in I'\}$ where $I' \subset I$. Then, since $\pi_{I'} : X \to X_{I'}$ is continuous, open and onto, $X_{I'}$ has the property $\bar{1}_{m,n}$.

3.2. Products of Two Spaces

We now investigate the productivity of weak m-n compactness via techniques of Stephenson and Vaughan [65].

A. Lemma. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a strong m-n stable filter base on X and let $\mathcal{G}$ be a strong m-n stable filter base on Y. Suppose every member of $\mathcal{F}$ meets every member of $f^{-1}(\mathcal{G})$. Then $\mathcal{F} \vee f^{-1}(\mathcal{G})$ is a strong m-n stable filter base on X.

Proof. We note that $|\mathcal{F} \vee f^{-1}(\mathcal{G})| \leq |\mathcal{F} \cup f^{-1}(\mathcal{G})| \leq m$. Let $\mathcal{H}$ be a sub-collection of $\mathcal{F} \vee f^{-1}(\mathcal{G})$ with $|\mathcal{H}| < n$. Let $H \in \mathcal{H}$; then $H = (\cap \mathcal{F}_H) \cap (f^{-1}(\mathcal{G}_H))$ for some $\mathcal{F}_H \subset \mathcal{F}$ and $\mathcal{G}_H \subset \mathcal{G}$ with $|\mathcal{F}_H| < \aleph_0$ and $|\mathcal{G}_H| < \aleph_0$. Let $\mathcal{F}' = \cup\{\mathcal{F}_H : H \in \mathcal{H}\}$ and $\mathcal{G}' = \cup\{\mathcal{G}_H : H \in \mathcal{H}\}$. Then $|\mathcal{F}'| < n$, $|\mathcal{G}'| < n$ and since $\mathcal{F}$ and $\mathcal{G}$ are $<n$-strongly stable, there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\mathrm{Int}(\cap \mathcal{F}') \supseteq F$ and $\mathrm{Int}(\cap \mathcal{G}') \supseteq G$. By the continuity of f we have

$$\begin{aligned}\mathrm{Int}(\cap \mathcal{H}) &= \mathrm{Int}(\cap \mathcal{F}') \cap \mathrm{Int}(f^{-1}(\cap \mathcal{G}')) \\ &\supseteq \mathrm{Int}(\cap \mathcal{F}') \cap f^{-1}(\mathrm{Int}(\cap \mathcal{G}')) \\ &\supseteq F \cap f^{-1}(G)\end{aligned}$$

and hence $\mathcal{F} \vee f^{-1}(\mathcal{G})$ is a strong m-n stable filter base on X.

B. Theorem. Let n be a regular cardinal and $m^{\underline{n}} = m$. Let X be a space which satisfies the property $\bar{1}_{m,n}$. If Y is weakly m-n compact, then $X \times Y$ is weakly m-n compact.

Proof. Let $\mathcal{F}$ be a strong m-n stable filter on $X \times Y$. Then, since $\pi_1 : X \times Y \to X$ is an open map, $\pi_1(\mathcal{F})$ is a strong m-n stable filter base on X and since X satisfies the property $\bar{1}_{m,n}$, there exists a strong m-n stable filter base $\mathcal{G}$ on X and a compact subset K of X such that $\mathcal{G} > \pi_1(\mathcal{F})$ and $\mathcal{G} > \mathcal{V}_k$. Let $\mathcal{H} = \mathcal{F} \vee \pi_1^{-1}(\mathcal{G})$, then by the lemma A, $\mathcal{H}$ is a strong m-n stable filter base on $X \times Y$. Again, since $\pi_2 : X \times Y \to Y$ is an open map, $\pi_2(\mathcal{H})$ is a strong m-n stable filter base on Y and since Y is weakly m-n compact, $\cap\overline{\pi_2(\mathcal{H})} \neq \Phi$. Let $y \in \cap \overline{\pi_2(\mathcal{H})}$ and let W be an open neighbourhood of y. Then we note $(G \times W) \cap F \neq \Phi$ for every $G \in \mathcal{G}$ and for every $F \in \mathcal{F}$. Hence $(\cap \bar{\mathcal{F}}) \cap K \times \{y\} \neq \Phi$ and therefore $X \times Y$ is weakly m-n compact.

C. Corollary. Let X and Y be weakly $m - \aleph_0$ compact spaces. If X has character $\leq m$ (or X is locally compact), then $X \times Y$ is weakly $m - \aleph_0$ compact.

Proof. Take $m = n = \aleph_0$ in proposition 1-B (or 1-C) and apply theorem B.

D. Remark. The restriction on the factor space X cannot be relaxed in C (see 3.5 of [59]).

E. Theorem. Let n be a regular cardinal and $m^{\bar{n}} = m$. Let X and Y be two spaces which satisfy the property $\bar{I}_{m,n}$. Then $X \times Y$ satisfies the property $\bar{I}_{m,n}$.

Proof. Let F be a strong m-n stable filter base on $X \times Y$. Then since X satisfies the property $\bar{I}_{m,n}$, there exists a strong m-n stable filter base G_1 on X and a compact subset K_1 of X such that $G_1 > \pi_1(F)$ and $G_1 > V_{k_1}$. Then by the lemma A, $H = F \vee \pi_1^{-1}(G_1)$ is a strong m-n stable filter base on $X \times Y$ and again, since Y satisfies $\bar{I}_{m,n}$, there exists a strong m-n stable filter base G_2 on Y and a compact subset K_2 of Y such that $G_2 > \pi_2(H)$ and $G_2 > V_{K_2}$. Then by the lemma A, $H \vee \pi_2^{-1}(G)$ is a strong m-n stable filter base on $X \times Y$ and we note that $H \vee \pi_2^{-1}(G_2) > F$ and $H \vee \pi_2^{-1}(G_2) > V_{K_1 \times K_2}$. Hence $X \times Y$ satisfies the property $\bar{I}_{m,n}$.

Thus the property $\bar{I}_{m,n}$ is finitely productive, provided n is regular and $m^{\bar{n}} = m$.

3.3. Countable Products

We shall extend the results of section 2 to countable products.

A. Theorem. Let $X = \Pi\{X_i : i = 1,2,\ldots\}$ and let n be a regular cardinal with $m^{\bar{n}} = m$. Suppose each X_i satisfies the property $\bar{I}_{m,n}$. Then X is weakly m-n compact.

Proof. Let F be a strong m-n stable filter base on X. Then $\pi_1(F)$ is a strong m-n stable filter base on X_1. Since X_1 satisfies the

property $\bar{I}_{m,n}$, there exists a strong m-n stable filter base $\mathcal{G}_1$ on X_1 and a compact subset $K_1 \subset X_1$ such that $\mathcal{G}_1 > \pi_1(\mathcal{F})$ and $\mathcal{G}_1 > \mathcal{V}_{K_1}$. Then by lemma 2-A, $\mathcal{H}_1 = \mathcal{F} \vee \pi_1^{-1}(\mathcal{G}_1)$ is a strong m-n stable filter base on X.

Now inductively assume that for each $i \leq k$, a compact $K_i \subset X_i$ and a strong m-n stable filter base $\mathcal{G}_i$ on X_i have been found such that

(i) $\mathcal{G}_i > \mathcal{V}_{K_i}$, and

(ii) $\mathcal{H}_k = \mathcal{F} \vee [\vee\{\pi_1^{-1}(\mathcal{G}_i): i = 1,2,\ldots,k\}]$ is a strong m-n stable filter base on X.

Since X_{k+1} satisfies the property $\bar{I}_{m,n}$, it is clear that there exists a compact $K_{k+1} \subset X_{k+1}$ and a strong m-n stable filter base $\mathcal{G}_{k+1}$ on X_{k+1} such that

(i) $\mathcal{G}_{k+1} > \mathcal{V}_{K_{k+1}}$, and

(ii) $\mathcal{H}_{k+1} = \mathcal{H}_k \vee \pi_{k+1}^{-1}(\mathcal{G}_{k+1})$ is a strong m-n stable filter base on X.

This completes the inductive definition of K_i and $\mathcal{H}_i$ for $i = 1,2,\ldots$.

Now set

$$K = \Pi\{K_i : i = 1,2,\ldots\}$$
$$\mathcal{H} = \mathcal{F} \vee [\vee\{\pi_i^{-1}(\mathcal{G}_i) : i = 1,2,\ldots\}].$$

Then K is a compact subset of X and $\mathcal{H}$ is a filter base on X, satisfying $\mathcal{H} > \mathcal{F}$ and $\mathcal{H} > \mathcal{V}_K$. Thus

$$\cap\bar{\mathcal{F}} \supset \cap\bar{\mathcal{H}} \supset (\cap\bar{\mathcal{H}}) \cap K$$

and, since K is compact, it follows that $(\cap\bar{\mathcal{H}}) \cap K \neq \phi$. Thus $\mathcal{F}$ has non-empty adherent in X and, by 2.1-I, X is weakly m-n compact.

B. <u>Corollary</u>. Let $X = \Pi\{X_i : i = 1,2,\ldots\}$ and let n be a regular cardinal with $\overline{m}^{\,n} = m$. Suppose each X_i is weakly m-n compact, <n-discrete and has character $\leq m$. Then X is weakly m-n compact.

4. Large Products

In general dense subspaces will not inherit the property weakly Lindelöf (relatively weakly Lindelöf). In this section we consider the following problem in a more general setting:

> Let $X = \Pi\{X_i : i \in I\}$. Let γ be an infinite cardinal. Is the property weakly Lindelöf in $\gamma(\Pi X_i)$ determined by finite sub-products of ΠX_i?

4.1. Machinery

We shall establish two special cases of the main theorem by considering γ-weak topological sums of $X = \Pi\{X_i : i \in I\}$.

A. Definition. Let $\mathcal{B}$ be a base for a topological space X. Then X is said to be $\mathcal{B}$-weakly m-n compact if and only if for every cover $\mathcal{U} \in P(\mathcal{B})$ of X with $|\mathcal{U}| = m$ there exists a $\mathcal{V} \in P(\mathcal{U})$ such that $|\mathcal{V}| < n$ and $X = \overline{\cup\mathcal{V}}$.

We note the following facts about $\mathcal{B}$-weakly m-n compact spaces:

(i) If X is weakly m-n compact then X is $\mathcal{B}$-weakly m-n compact.

(ii) X is $\mathcal{B}$-weakly ∞-n compact if and only if X is weakly ∞-n compact.

B. Proposition. Let n and γ be infinite cardinals such that $n \geq \gamma$. Let $\overline{\gamma} = \gamma$ if γ is regular and let $\overline{\gamma} = \gamma^{+}$ if γ is singular. If n is regular, then $\overline{\gamma} \leq n$.

Proof. (i) If γ is regular, then $\bar{\gamma} = \gamma$ and hence $\bar{\gamma} \leq n$.

(ii) If γ is singular, then since n is regular, we have $\gamma < n$. Hence $\bar{\gamma} = \gamma^{+} \leq n$.

C. Remark. $\bar{\gamma}$ in the proposition B is a regular cardinal.

D. Notation. Let I be any indexing set. Then we denote,

(i) $|I|^{\underline{\gamma}} = \sum\{|I|^{k} : k < \gamma\}$ and

(ii) $P_{<\gamma}(I) = \{I' \subset I : |I'| < \gamma\}$

E. Lemma. Let $X = \Pi\{X_i : i \in I\}$ and let $m \geq n \geq cf(n) > |I|^{\underline{\gamma}} \geq \gamma \geq k \geq \aleph_0$. Suppose $(X_{I'})_k$ is weakly m-n compact for all $I' \in P_{<\gamma}(I)$. Then $\gamma(\Pi X_i)$ is weakly m-n compact relative to $(\Pi X_i)_k$.

Proof. We note that $\gamma(\Pi X_i) = \cup\{X(I') : I' \in P_{\gamma}(I)\}$ and since $X(I')$ is homeomorphic to $(X_{I'})_k$, $\gamma(\Pi X_i)$ is the $|I|^{\underline{\gamma}}$- fold union of weakly m-n compact subspaces of $(\Pi X_i)_k$. Hence we have the lemma by 1.1 E.

The sets $W = \Pi\{W_i : i \in I\}$ where each W_i is open in X_i and $|R(W)| < k$ form the canonical basis for $(\Pi X_i)_k$. Let $A \subset \Pi\{X_i : i \in I\}$; then the canonical basis for A consists of all sets of the form $A \cap W$ with W as above. In this terminology we rephrase the lemma E as follows:

F. Lemma. Let $X = \Pi\{X_i : i \in I\}$ and let $m \geq n \geq cf(n) > |I|^{\underline{\gamma}} \geq \gamma \geq k \geq \aleph_0$.

Suppose $(X_{I'})_k$ is weakly m-n compact with respect to its canonical basis for all $I' \in P_{<\gamma}(I)$. Then $\gamma(\Pi X_i)$ is weakly m-n compact with respect to its canonical basis.

G. Theorem. Let $X = \Pi\{X_i : i \in I\}$ and let $m \geq n > |I| \geq \gamma \geq k \geq \aleph_0$. Suppose n is regular and strongly γ-inaccessible and suppose $(X_{I'})_k$ is weakly m-n compact for all $I' \in P_{<\gamma}(I)$. Then $\gamma(\Pi X_i)$ is weakly m-n compact relative to $(\Pi X_i)_k$.

Proof. Consider $|I|^{\underline{\gamma}} = \Sigma\{|I|^k : k < \gamma\} \geq |I| \geq \gamma$ and since n is regular and strongly γ-inaccessible we have $n = cf(n)$ and $|I|^{\underline{\gamma}} < n$. Hence $m \geq n = cf(n) > |I|^{\underline{\gamma}} \geq \gamma \geq k \geq \aleph_0$ and therefore we can apply lemma E to obtain the theorem.

In the above theorem there is a restriction on the cardinality of I and we wish to relax this condition in the next section.

4.2. Main Theorem

We establish the main theorem about products of weakly m-n compact spaces, by employing techniques similar to those used by Comfort in [13].

A. Theorem. Let $m \geq n > \gamma \geq k \geq \aleph_0$ and let n be regular and strongly γ-inaccessible. Let $X = \Pi\{X_i : i \in I\}$ and let $\mathcal{U} \subseteq \{U = \Pi(U_i : i \in I\}$: each U_i is open in X_i and $|R(U)| < k\}$ where $|\mathcal{U}| = m$. Suppose $(X_{I'})_k$ is weakly m-n compact for all $I' \in P_{<\gamma}(I)$ and $\gamma(\Pi X_i) \subseteq \cup\mathcal{U}$. Then there exists a $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| < n$ and $\gamma(\Pi X_i) \subseteq \overline{\cup\mathcal{U}'}$.

<u>Proof</u>. Let $\bar{\gamma} = \gamma$ if γ is regular and $\bar{\gamma} = \gamma^+$ if γ is singular. Then $\bar{\gamma}$ is regular and $\bar{\gamma} \leq n$. By theorem 1-G, $\gamma(X_{I'})$ is weakly m-n compact relative to $(X_{I'})_k$, for all $I' \in P_{<\gamma}(I)$. We note that $\pi_{I'}(\gamma(\Pi X_i)) = \gamma(X_{I'})$ and hence $\{\pi_{I'}(U) : U \in \mathcal{U}\}$ is an m-fold open cover of $\gamma(X_{I'})$ where $I' \subset I$. Let $I' \in P_{<\bar{\gamma}}(I)$; then there exists a $\mathcal{U}_{I'} \subset \mathcal{U}$ such that $|\mathcal{U}_{I'}| < n$ and

$$\gamma(X_{I'}) \subseteq \mathrm{Cl}_{(X_{I'})_k}(\cup\{\pi_{I'}(U) : U \in \mathcal{U}_{I'}\}) . \tag{1}$$

Let $I_1 \subset I$ and $|I_1| < n$ and let $\mathcal{F}_1 = \{\mathcal{U}_{I'} : I' \in P_{<\bar{\gamma}}(I_1)$ and $\mathcal{U}_{I'}$ has the property (1) . Let $I_2 = I_1 \cup \mathcal{R}(\mathcal{F}_1)$ where $\mathcal{R}(\mathcal{F}_1) = \cup\{\mathcal{R}(U) : U \in \mathcal{U}_{I'}$ and $\mathcal{U}_{I'} \in \mathcal{F}_1\}$. Trivially $\mathcal{R}(\mathcal{F}_1) \subset I$ and therefore $I_2 \subset I$.

We note the following:

(i) $|\mathcal{R}(U)| < k \leq n$.

(ii) $\mathcal{U}_{I'} \subset \mathcal{U}$, $|\mathcal{U}_{I'}| < n$ for all $I' \in P_{<\bar{\gamma}}(I_1)$.

(iii) $|P_{<\bar{\gamma}}(I_1)| \leq |I_1|^{\underline{\gamma}}$ if γ is regular and $|P_{<\bar{\gamma}}(I_1)| \leq |I_1|^{\underline{\gamma^+}}$ if γ is singular.

Since n is strongly γ-inaccessible, $|P_{<\bar{\gamma}}(I_1)| < n$ for all $\gamma \leq n$. Hence we have the following:

$$\begin{aligned} &\text{(i)} \quad |\mathcal{R}(\mathcal{F}_1)| < n, \\ &\text{(ii)} \quad |I_2| < n. \end{aligned} \tag{2}$$

Inductively we define $\mathcal{F}_\alpha = \cup\{\mathcal{U}_{I'} : I' \in P_{<\bar{\gamma}}(I_\alpha)\}$ and $I_{\alpha+1} = I_\alpha \cup \mathcal{R}(\mathcal{F}_\alpha)$ for $\alpha < \bar{\gamma}$. Let $I^* = \cup\{I_\alpha : \alpha < \bar{\gamma}\}$ and $\mathcal{U}' = \cup\{\mathcal{F}_\alpha : \alpha < \bar{\gamma}\}$. Since n is regular and $|I_\alpha| < n$ for all $\alpha < \bar{\gamma}$ we have

$$\begin{aligned} &|I^*| < n \quad \text{and} \\ &|\mathcal{U}'| < n\,. \end{aligned} \tag{3}$$

Each $\mathcal{U}_{I'} \subset \mathcal{U}$ and therefore each $\mathcal{F}_\alpha \subset \mathcal{U}$ and hence $\mathcal{U}' \subset \mathcal{U}$. We shall prove that

$$\gamma(\Pi X_i) \subseteq \overline{\cup\mathcal{U}'}\,.$$

Let $x \in \gamma(\Pi X_i)$ and let $V = \Pi\{V_i : i \in I\}$ be a basic open neighbourhood of x in $(\Pi X_i)_k$. Then we have $|\mathcal{R}(V)| < k \leq \gamma \leq \overline{\gamma}$ and hence there exists an $\alpha < \overline{\gamma}$ such that

$$\mathcal{R}(V) \cap I^* = \mathcal{R}(V) \cap I_\alpha\,. \tag{4}$$

Let $H = \mathcal{R}(V) \cap I_\alpha$; then $H \subset I_\alpha \subset I$ and $|H| < k < \overline{\gamma}$ and, by (1), there exists a $U \in \mathcal{U}_H \subset \mathcal{F}_\alpha$ such that

$$\pi_H(U) \cap \pi_H(V) \neq \Phi\,. \tag{5}$$

Since $U \in \mathcal{F}_\alpha$, $\mathcal{R}(U) \subset I_{\alpha+1}$ and, by (4),

$$\begin{aligned} \mathcal{R}(U) \cap \mathcal{R}(V) &= (\mathcal{R}(U) \cap I_{\alpha+1}) \cap \mathcal{R}(V) \\ &= \mathcal{R}(U) \cap (I_{\alpha+1} \cap I^*) \cap \mathcal{R}(V) \\ &= \mathcal{R}(U) \cap (\mathcal{R}(V) \cap I^*) \cap I_{\alpha+1} \\ &= \mathcal{R}(U) \cap \mathcal{R}(V) \cap I_\alpha \\ &\subseteq H\,. \end{aligned} \tag{6}$$

Now $U \cap V \neq \Phi$ and therefore $V \cap (\cup\mathcal{U}') \neq \Phi$. This is true for every neighbourhood V of x and therefore we have $\gamma(\Pi X_i) \subseteq \overline{\cup\mathcal{U}'}$.

For $k \leq \gamma$, $\gamma(\Pi X_i)$ is a dense subspace of $(\Pi X_i)_k$ and we are ready to give the main theorem.

B. <u>Theorem</u>. Let $m \geq n > \gamma \geq k \geq \aleph_0$ and let n be regular and strongly γ-inaccessible. Let $X = \Pi\{X_i : i \in I\}$ and suppose $(X_{I'})_k$ is weakly m-n compact for all $I' \in P_{<\gamma}(I)$. Let $\mathcal{B}$ be the canonical base for the product space $(\Pi X_i)_k$. Then $(\Pi X_i)_k$ is $\mathcal{B}$-weakly m-n compact.

If n is strongly γ-inaccessible and if $d(X_i) < n$ for each $i \in I$, then each X_i has a dense subset A_i with $|A_i| < n$. Hence $A_{I'}$ is a dense subset of $(X_{I'})_k$ and $|A_{I'}| < n$ for all $I' \in P_\gamma(I)$ where $\gamma \leq n$ and $A = \Pi\{A_i : i \in I\}$. Therefore $d(X_{I'}) < n$ and hence $(X_{I'})_k$ is weakly ∞-n compact for all $I' \in P_\gamma(I)$. Thus we have the following:

C. <u>Corollary</u>. Let $X = \Pi\{X_i : i \in I\}$, let $n \geq \gamma \geq k \geq \aleph_0$ and suppose n is regular and strongly γ-inaccessible. If $d(X_i) < n$ for all $i \in I$, then $(\Pi X_i)_k$ is weakly ∞-n compact.

4.3. <u>Weakly ∞-n Compact Spaces</u>

We recall that $\mathcal{B}$-weak ∞-n compactness is equivalent to weak ∞-n compactness and hence we obtain product theorems for weakly ∞-n compact spaces as special cases of theorem 2-B (see [76]).

A. <u>Theorem</u>. Let $X = \Pi\{X_i : i \in I\}$ and let $n > \gamma \geq k \geq \aleph_0$. Suppose n is regular and strongly γ-inaccessible. Then $(\Pi X_i)_k$ is weakly ∞-n compact if and only if $(X_{I'})_k$ is weakly ∞-n compact for all $I' \in P_\gamma(I)$.

B. Corollary. Let $X = \Pi\{X_i : i \in I\}$ with the usual product topology and let n be a regular cardinal. Then X is weakly ∞-n compact if and only if every finite sub-product of X is weakly ∞-n compact.

Proof. Necessity follows from the fact that $\pi_{I'} : X \to X_{I'}$ is continuous for every $I' \subset I$.

On the other hand, note that regular cardinals are infinite and every infinite cardinal is strongly $\aleph_0$-inaccessible. Hence taking $\gamma = k = \aleph_0$ in theorem A, we obtain sufficiency.

Let $X = \Pi\{X_i : i \in I\}$. Then, according to theorem 2.3 A, if each X_i is weakly ∞-n compact and $<n$-discrete (or each X_i is weakly ∞-n compact and locally compact) where n is regular, then $X_{I'}$ is weakly ∞-n compact, for all $I' \in P_{<\aleph_1}(I)$.

Now we are ready to produce two theorems regarding weakly ∞-n compact spaces.

C. Theorem. Let $X = \Pi\{X_i : i \in I\}$ and let n be a regular cardinal. Suppose each X_i is $<n$-discrete. Then X is weakly ∞-n compact if and only if each X_i is weakly ∞-n compact.

D. Theorem. Let $X = \Pi\{X_i : i \in I\}$ and let n be an uncountable regular cardinal. Suppose each X_i is locally compact. Then X is weakly ∞-n compact if and only if each X_i is weakly ∞-n compact.

Since $\aleph_0$ and $\aleph_1$ are regular cardinals taking $n = \aleph_0$ and $n = \aleph_1$ respectively in C, yield the following results:

E. Corollary.

(i) A non-empty product is H-closed if and only if each factor space is H-closed (see [39]).

(ii) A non-empty product of P-spaces is weakly Lindelöf if and only if each factor space is weakly Lindelöf.

F. Note. Let Y be the weakly Lindelöf P-space constructed in 1.2 G. Then Y^k $(k > C)$ has the following properties:

(i) weakly Lindelof and T_2,

(ii) non-separable,

(iii) Non-cellular,

(iv) non-normal.

Taking $n = \aleph_1$ in theorem D, we obtain the following:

G. Theorem. Let $X = \Pi\{X_i : i \in I\}$. Suppose each X_i is locally compact. Then X is weakly Lindelöf if and only if each X_i is weakly Lindelöf.

CHAPTER IV: CARDINAL INVARIANTS

1. The Almost Lindelöf Number

1.1. Basic Results

In this section, we will introduce a new cardinal function, the almost Lindelöf number $aL(X)$ of a topological space X.

A. Definition. A subset E of a topological space X is said to be almost k-Lindelöf if and only if every X-open cover $\mathcal{U}$ of E has a sub-system $\mathcal{U}'$ of cardinality $\leq k$ with $E \subseteq \cup\{\overline{U} : U \in \mathcal{U}'\}$. We now define,

$$aL(E,X) = \min\{k : E \text{ is almost } k\text{-Lindelöf}\}$$

and define the almost Lindelöf number $aL(X)$ of X as:

$$aL(X) = \sup\{aL(E,X) : E \text{ is a closed subset of } X\} + \omega$$

B. Definition. A topological space X is said to be weakly k-Lindelöf if and only if every open cover $\mathcal{U}$ of X has a sub-system $\mathcal{U}'$ of cardinality $\leq k$ with $X = \overline{\cup\mathcal{U}'}$. Note that this is equivalent to saying that X is weakly ∞-k^+ compact.

We define the weak Lindelöf number $wL(X)$ of X as,

$$wL(X) = \min\{k : X \text{ is weakly } k\text{-Lindelöf}\} + \omega \quad \text{(see [4])}$$

C. Remark. For any topological space X, $wL(X) \leq aL(X) \leq L(X)$ and for regular spaces, $aL(X) = L(X)$.

We next present examples which show these cardinal functions will in general, differ.

D. Example. Let X be the closed upper half plane in $\mathbb{R}^2$ and let $E \subset X$ be the x-axis. The basic neighbourhoods of $x \in X - E$ will be as usual in $\mathbb{R}^2$, while the basic neighbourhoods of points $Z \in E$ will take the form

$$V_\varepsilon = \{x \in X - E : ||X - Z|| < \varepsilon\} \cup \{Z\} .$$

Call this space Y.

Easily, Y is a $T_{2\frac{1}{2}}$ (and thus T_2) space in which E is a discrete closed subspace of cardinality $\mathbb{C}$. It follows that $L(Y) = \mathbb{C}$. To see that $aL(Y) = \aleph_0$, it will be enough to note that $aL(B,Y) = \aleph_0$, for any $B \subseteq E$, since the open upper half plane in Y is hereditarily Lindelöf and whenever A and B are disjoint in Y, $aL(A,Y) + aL(B,Y) \geq aL(A \cup B,Y)$. But if $\mathcal{U}$ is a Y-open cover of $B \subseteq E$, and if R denotes the set E with the usual line topology, then

$$\{\mathrm{Int}_R[E \cap \mathrm{Cl}_Y U] : U \in \mathcal{U}\}$$

is an open cover of B in $\mathbb{R}$, which is hereditarily Lindelöf.

Thus, in non-regular spaces, we may find $aL(X) < L(X)$.

Now, let V be the T_2 space constructed in 3.1.2 C. For this space V,

$$aL(V) = k > n > wL(V).$$

Thus we see that, in general, the cardinal functions $wL(X)$, $aL(X)$ and

$L(X)$ are distinct in T_2 spaces.

For notational convenience we use $\overline{\psi}(X)$ for $\psi_C(X)$ (page 8, [34]).

E. Lemma. Let X be a T_2-space. Then $|X| \leq d(X)^{L(X)\psi(X)\partial(X)}$.

Proof. Let B be a dense subset of X such that, $|B| \leq d(X)$. Let $k = L(X)\psi(X)\partial(X)$. Since $\partial(X) \leq k$, $X = \cup\{\bar{T} : T \subseteq B, |T| \leq k\}$ and by 2.2 of [9], it follows that

$$\begin{aligned} |\bar{T}| &\leq \rho(\bar{T})^{aL(\bar{T})\psi(\bar{T})} \\ &\leq (2^{d(\bar{T})})^{aL(\bar{T})\psi(\bar{T})} \\ &\leq 2^k. \end{aligned}$$

Hence,

$$\begin{aligned} |X| &\leq 2^k \cdot |B|^k \\ &\leq 2^k (d(X))^k \\ &= d(X)^{L(X)\psi(X)\partial(X)} \end{aligned}$$

F. Theorem. Let X be a T_2-space. Then $|X| \leq 2^{L(X)\psi(X)\partial(X)}$.

Proof. Let $k = L(X)\psi(X)\partial(X)$. For each $x \in X$, we select a family $\mathcal{O}_x$ of neighbourhoods of x such that, $|\mathcal{O}_x| \leq k$ and $\cap\mathcal{O}_x = \{x\}$. By transfinite induction we define a sequence of closed subsets of X such that,

$$F_1 \subset F_2 \subset \ldots \subset F_\xi \subset \ldots \subset X$$

where $|F_\xi| \leq 2^k$ for each $\xi < k^+$ and for each $1 \leq \xi < k^+$, if

$$\mathcal{U} \subset \cup\{\mathcal{O}_x : x \in \cup\{F_\alpha : \alpha < \xi\}\}$$

where $|\mathcal{U}| \leq k$ and $X - \cup\mathcal{U} \neq \Phi$, then $F_\xi - \cup\mathcal{U} \neq \Phi$. Now let $F_1 = \{p\}$, where p is an arbitrary point of X. Suppose we have defined F_α, for $\alpha < \xi$, with $|F_\alpha| \leq 2^k$. Let

$$\mathcal{O} = \cup\{\mathcal{O}_x : x \in \bigcup_{\alpha<\xi} F_\alpha\}$$

and let $\mathcal{O}^* = \{X - \cup\mathcal{U} : \mathcal{U} \in \mathcal{O}$ and $|\mathcal{U}| \leq k\}$. We select one point from each non-empty set in $\mathcal{O}^*$ and form the set E.

We take $F_\xi = \overline{E \cup (\cup\{F_\alpha : \alpha < \xi\})}$. Then F_ξ is closed and by lemma E, $|F_\xi| \leq 2^k$. Now, we note that, $X = \cup\{F_\xi : \xi < k^+\}$ and thus we are done.

G. <u>Corollary</u> (A.V. Arhangelskii, 1970). Let X be a T_2-space. Then $|X| \leq 2^{L(X)\chi(X)}$.

H. <u>Theorem</u>. Let X be a T_2-space. Then $|X| \leq d(X)^{\pi\chi(X)\bar{\psi}(X)}$.

<u>Proof</u>. Let B be a dense subset of X such that $|B| \leq d(X)$ and for each $p \in X$ let $\mathcal{V}_p$ be a local π-base at p with $|\mathcal{V}_p| \leq \pi\chi(X)$. Let $k = \pi\chi(X)\bar{\psi}(X)$. Choose $q(V) \in V \cap B$ for each $V \in \mathcal{V}_p$, and let

$$B_p = \{q(V) : V \in \mathcal{V}_p\}.$$

Then $B_p \in [B]^{\leq k}$ for each $p \in X$. Let $\mathcal{K}_p$ be a collection of neighbourhoods of p such that $|\mathcal{K}_p| \leq k$ and

$$\{p\} = \cap\{\bar{K} : K \in \mathcal{K}_p\}.$$

Then for each $K \in \mathcal{K}_p$, $p \in \overline{K \cap B_p}$. Hence

$$\{p\} = \cap\{\overline{K \cap B_p} \cdot K \in \mathcal{K}_p\}.$$

Now the map $p \to \{K \cap B_p : K \in \mathcal{K}_p\}$ from X to the collection $[[B]^{\leq k}]^{\leq k}$ is one to one. Thus,

$$\begin{aligned} |X| &\leq (d(X)^k)^k \\ &= d(X)^{\pi\chi(X)\bar{\psi}(X)} . \end{aligned}$$

I. Corollary.

(i) Let X be a T_2-space. Then $|X| \leq d(X)^{\chi(X)}$ (see 2.5 of [34]).

(ii) Let X be a T_3-space. Then $|X| \leq d(X)^{\pi\chi(X)\psi(X)}$.

1.2. Main Results

We shall show that for a T_2-space X, $|X| \leq 2^{aL(X)\chi(X)}$ which improves the celebrated theorem (2-G) of A.V. Arhangelskii (1970).

First we note a purely set theoretic result.

A. Proposition. Let X be a set such that $|X| > k > \lambda$ and $k^\lambda = k$. Let $G : P_{\leq\lambda}(X) \to P_{\leq k}(X)$ be a set mapping. Then there exists a set $A \subseteq X$ such that $|A| = k$ and $A \supseteq G(B)$ for every $B \in P_{\leq\lambda}(A)$.

Proof. We note that,

$$\begin{aligned} k^{\underline{\lambda}^+} &= \sum\{k^\beta : \beta < \lambda^+\} \\ &= k \cdot \lambda^+ = k. \end{aligned}$$

Now we apply 2.24 of [34].

B. Theorem. Let X be a T_2-space. Then $|X| \leq 2^{aL(X)\bar{\psi}(X)\pi\chi(X)\partial(X)}$.

Proof. Let $\beta = aL(X)\bar{\psi}(X)\pi\chi(X)\partial(X)$, and let $k = 2^{\beta}$. Let $\mathcal{O}_x$ be a collection of open neighbourhood of x such that $|\mathcal{O}_x| \leq \beta$ and $\{x\} = \cap\{\bar{V} : V \in \mathcal{O}_x\}$. We shall write, $\mathcal{U}_A = \cup\{\mathcal{O}_x : x \in A\}$ and let G be a set mapping,

$$P_{\leq\beta}(X) \to P_{\leq k}(X).$$

Let $A \in P_{\leq\beta}(X)$; then we set

$$\mathcal{V}_A = \{\mathcal{U} \in P_{\leq\beta}(\mathcal{U}_A) : X - \cup\{\bar{U} : U \in \mathcal{U}\} \neq \Phi\}.$$

Now it is clear that $|\mathcal{V}_A| \leq 2^{\beta} = k$. For each $\mathcal{U} \in \mathcal{V}_A$, we select $p(\mathcal{U}) \in X - \cup\{\bar{U} : U \in \mathcal{U}\}$ and we write $G(A) = \overline{A \cup (\cup\{p(\mathcal{U}) : \mathcal{U} \in \mathcal{V}_A\})}$. Since $\pi\chi(X)\overline{\psi}(X) \leq \beta$, by 1-H.

$$\begin{aligned} |G(A)| &\leq (2^{\beta})^{\beta} \\ &= 2^{\beta} \\ &= k. \end{aligned}$$

Also, we note that $G(A) \supseteq \bar{A}$ for every $A \in P_{\leq\beta}(X)$.

We now apply Proposition A to obtain a set $B \subseteq X$ with $|B| = k$ and $B \supseteq G(A)$ for every $A \in P_{\leq\beta}(B)$. We claim that $X = B$.

First, since $\partial(X) \leq \beta$, it follows that B is a closed subset of X. Now suppose $X - B \neq \Phi$, say $q \in X - B$. For each $y \in B$, we select a $V_y \in \mathcal{O}_y$ such that $q \notin \bar{V}_y$ and, since $aL(B,X) \leq \beta$, there exists a

$Y \in P_{\leq\beta}(B)$ such that $B \subseteq \cup\{\bar{V}_y : y \in Y\} \subseteq X - \{q\}$. Thus $\mathcal{U} = \{V_y : y \in Y\} \in \mathcal{V}_Y$ and by the construction, $p(\mathcal{U}) \in G(Y) \subseteq B$. But $p(\mathcal{U}) \in X - \cup\{\bar{V}_y : y \in Y\} \subseteq X - B$. Hence, we have a contradiction.

C. Corollary. Let X be a T_2-space. Then $|X| \leq 2^{aL(X)\chi(X)}$.

Following the main lines of the proof of the theorem B, with a slight modification, we obtain the following result:

D. Theorem. Let X be a T_2-space. Then

$$|X| \leq 2^{aL^*(X)\bar{\psi}(X)}.$$

We shall extend the above result, to obtain an upper bound for the cardinality of the family $\mathcal{K}(X)$ of compact subsets of a T_2-space X.

E. Definition. We denote the pseudocharacter of a subset C of X by $\Psi(C,X)$. Then the compact pseudocharacter $\Psi_K(X)$ of X is defined as

$$\Psi_K(X) = \sup\{\Psi(C,X) : C \text{ is a compact subset of } X\}.$$

It is clear that $\psi(X) \leq \Psi_K(X)$ and we shall show that, $\Psi_K(X) \leq aL^*(X)$.

F. Proposition. Let X be a T_2-space. Then

$$\Psi_K(X) \leq aL^*(X).$$

Proof. Let $aL^*(X) = k$. Let C be a compact subset of X and let p be a point in $X-C$. Then we can find two disjoint open subsets U_p and V_p such that, $p \in U_p$ and $C \subset V_p$. Thus $\bar{U}_p \cap C = \Phi$. Now since

$aL(X-C,X) \leq k$, there exists a $B \in [X-C]^{\leq k}$ such that $X - C \subseteq \cup\{\bar{U}_p : p \in B\}$. Hence $C \supseteq \cap\{X-\bar{U}_p : p \in B\}$ and since $C \subseteq X - \bar{U}_p$ for every $p \in B$. We have $\Psi(C,X) \leq k$. From this it follows that,

$$\Psi_K(X) \leq k.$$

G. Theorem. Let X be a T_2-space. Then $|\mathcal{K}(X)| \leq 2^{aL^*(X)\bar{\psi}(X)}$.

Proof. Let $k = aL^*(X)\bar{\psi}(X)$. For any $\{p,q\} \in [X]^2$ with $p \neq q$, select disjoint open sets $U_{p,q}$ and $V_{p,q}$ such that $p \in U_{p,q}$ and $q \in V_{p,q}$. Let $\mathcal{B}$ be the family of all finite intersections formed by sets of the form $V_{p,q}$. Then by theorem D, $|\mathcal{B}| \leq |X| \leq 2^k$. Let K be a compact subset of X and let $p \notin K$. Then there exists a $B_p \in \mathcal{B}$ such that $p \in B_p \subset \bar{B}_p \subset X - K$. Therefore, if F is a closed subset of $X - K$, then $\mathcal{G} = \{B_p : p \in F\}$ is an open cover of F and, since $aL(F,X) \leq k$, there exists a $\mathcal{G}' \in [\mathcal{G}]^{\leq k}$ such that

$$F \subseteq \cup\{\bar{B}_p : B_p \in \mathcal{G}'\} = A(\bar{\mathcal{G}}') \subseteq X - K.$$

Now we recall that $\Psi_K(X) \leq k$, and hence $X - K = \cup\{F_\alpha : \alpha < k\}$, where each F_α is closed. Thus we can write $X - K = \cup\{A(\bar{\mathcal{G}}') : \alpha < k\}$ and since there are at most 2^k $A(\bar{\mathcal{G}}')$ sets, it follows that

$$\begin{aligned} |\mathcal{K}(X)| &\leq (2^k)^k \\ &= 2^k. \end{aligned}$$

H. Proposition. Let X be a T_2-space. Then $\bar{\psi}(X) \leq L^*(X)$.

Proof. Let $x \in X$ and let $\mathcal{V}_x$ be a collection of open neighbourhoods

of x such that $\{x\} = \cap\{\bar{V} : V \in \mathcal{V}_x\}$. Then there exists a sub-collection $\mathcal{B}_x$ of $\mathcal{V}_x$ such that $|\mathcal{B}_x| \leq L^*(X)$ and

$$X - \{x\} \subseteq \cup\{X - \bar{V} : V \in \mathcal{B}_x\}$$

Hence, $\{x\} = \cap\{\bar{V} : V \in \mathcal{B}_x\}$. Thus $\bar{\psi}(x,X) \leq L^*(X)$. Since $x \in X$ is arbitrary, it follows that

$$\bar{\psi}(X) \leq L^*(X).$$

I. Remark. We conclude this section by noting that, Theorem G simultaneously generalizes two important theorems 2.1 and 2.7 of [5].

Proof. We apply theorem G and proposition H.

In the next section we shall show that, the cardinal invariant $aL(X)$ is better than $L(X)$ in respect of estimations of the cardinality of X and the cardinality of $\mathcal{K}(X)$, for T_2-spaces.

1.3. Examples

We shall construct an example to show that there are T_2-spaces where $aL(X)$ is relatively small compared to $L(X)$ and $c(X)$.

A. Method. Let T be the product of the k copies of the unit interval I. Then T has a closed nowhere dense subset E of cardinality 2^k. Let X be the set T with topology described as follows. The neighbourhoods of points $p \in T - E$ will be unchanged in X, while neighbourhoods of points $p \in E$ will take the form

$$U_p^* = (U-E) \cup \{p\},$$

where U is a neighbourhood of p in T.

Clearly, X is a T_2-space and since E is a closed discrete subset of X of cardinality 2^k, $L(X) = 2^k$. We shall show that $aL^*(X) = k$.

B. Lemma. Let T and X be the spaces mentioned in A. Let U be an open subset in T with $p \in U \cap E$. Then $C\ell_X U_p^* \supseteq U$.

Proof. Let $x \in U$. If $x \notin E$, then $x \in U_p^*$ and therefore we assume that $x \in U \cap E$. Let V_x^* be a neighbourhood of x in X. Then $V_x^* = (V - E) \cup \{x\}$ where V is open in T. Then U and V are both neighbourhoods of x in T and hence $U \cap V$ is a neighbourhood of x in T. Since E is nowhere dense, we must have $(U \cap V) - E \neq \Phi$. Now, it follows that $V_x^* \cap U_p^* \neq \Phi$. Thus $x \in C\ell_X U_p^*$. This proves the lemma.

C. Proposition (5.3 of [34]). Let $R = \Pi\{R_i : i \in I\}$, where each R_i is a T_1-space. Then

(i) $w(R) = |I| \cdot w_I(R)$

(ii) $\psi(R) = |I| \cdot \psi_I(R)$

Now, $L^*(T) \leq k$ and $\psi(T) = k$, where T is the space mentioned in A. Thus, using the space X of A, we establish the following:

D. Proposition. For each cardinal $k \geq \aleph_0$, there is a T_2-space X with,

(i) $aL^*(X) = k$

(ii) $L(X) = 2^k$

(iii) $\chi(X) = k$.

Proof. The space X constructed in A will serve. We note that, $|X| = 2^k$; in fact according to 3-G, $|K(X)| = 2^k$.

Our next aim is to show that there are T_2-spaces X with $L(X)$ and $c(X)$ both larger than $aL(X)$.

E. Proposition. For each cardinal $k \geq \aleph_0$, there is a T_2-space Y with,

(i) $aL(Y) \leq k$

(ii) $L(Y) = 2^k$

(iii) $c(Y) = 2^k$

Proof. Let X_1 be the space constructed via T in A. Let X_2 be the Alexandroff double of T. Then we take Y to be the topological union of X_1 and X_2. Since X_2 is an open subspace of Y and $c(X_2) = 2^k$, it follows that $Y = X_1 \oplus X_2$ has the required properties.

It is also clear that $\chi(Y) = k$ and $|Y| = 2^k$.

2. The Weak Lindelöf Number

Our main objective is to investigate the following problem:

> Let X be a T_3-space. Let $wL(X)$ denote the weak Lindelöf number of X and let $\chi(X)$ denote the character of X. Is,
>
> $$|X| \leq 2^{wL(X)\chi(X)} ?$$

2.1. Related Results

We shall define a new cardinal function, the quasi Lindelof number, which we shall denote by $qL(X)$, and we show that $qL(X)$ is a common lower bound for $aL(X)$ and $c(X)$.

A. Definition. A subset E of X is said to be relatively weakly k-Lindelof if and only if every X-open cover $\mathcal{U}$ of E has a subsystem $\mathcal{U}'$ of cardinality $\leq k$ with $E \subseteq \overline{\cup\mathcal{U}'}$.

Note that this is equivalent to saying that E is relatively weakly ∞-k^+ compact.

B. Definition. We define the relative weak Lindelöf number $RwL(E)$ of $E \subset X$ as, $RwL(E) = \min\{k : E$ is relatively weakly k-Lindelöf$\}$. Now we define the quasi Lindelöf number $qL(X) = \sup\{RwL(E) : E$ is a closed subset of $X\} + \omega$.

For every closed subset E of X, $RwL(E) \leq aL(X)$. Thus it is clear that, $wL(X) \leq qL(X) \leq aL(X) \leq L(X)$.

By the next proposition it will follow that $qL(X) \leq c(X)$.

C. Proposition. Let X be any topological space. Then the following are equivalent:

(i) $c(X) \leq n$,

(ii) if $\mathcal{U}$ is a collection of open subsets of X such that $X = \overline{\cup\mathcal{U}}$, then there exists a $\mathcal{U}' \subseteq \mathcal{U}$ such that $|\mathcal{U}'| \leq n$ and $X = \overline{\cup\mathcal{U}'}$,

(iii) if $\mathcal{U}$ is any collection of open subsets of X, then there exists a $\mathcal{U}' \subseteq \mathcal{U}$ such that, $|\mathcal{U}'| \leq n$ and $\cup\mathcal{U} \subseteq \overline{\cup\mathcal{U}'}$.

Proof. See 3.2 of [9].

D. Example. Let X be the product of $\aleph_1$ copies of the natural numbers $\mathbb{N}$. Let Y be the Alexandroff double of the unit interval I. We take V as the topological sum $X \oplus Y$. Then V is a $T_{3\frac{1}{2}}$ space and by c, $qL(X) = c(X) = \aleph_0$ and since $qL(V) \leq qL(X) + qL(Y)$, we conclude

(i) $qL(V) = \aleph_0$

(ii) $L(V) = \aleph_1$.

Since, in normal spaces $qL(X) = wL(X)$ we state a refinement of Juhász's theorem 2.36 [34], in the following manner:

E. Theorem. Let X be a T_3 space. Then $|X| \leq 2^{\chi(X)qL(X)}$.

2.2. Π-Normal Spaces

We shall introduce a new class of spaces, the Π-normal (ΠN) spaces, and we prove that $|X| \leq 2^{wL(X)\chi(X)}$ for all $X \in \Pi N$ where $T_4 \subsetneq \Pi N \subsetneq T_3$. This extends the result 2.36 of [34].

A. Definition. Let $a \in X$. A local π-base at a in X is a family $\mathcal{U}_a$ of proper, non-empty open subsets of X such that every neighbourhood of a contains a member of $\mathcal{U}_a$.

A normal local π-base is a local π-base with the property that the closures of the members are normal subspaces.

B. Definition. A T_3-space X is called Π-normal (ΠN) if and only if X has a normal local π-base $\mathcal{U}_a$ for every $a \in D$, where D is some dense subset of X.

C. Note. The following classes of spaces belong to the class ΠN:

(i) locally-metrizable and T_3,

(ii) locally paracompact and T_3,

(iii) locally normal and T_3,

(iv) locally Lindelöf and T_3.

In particular every T_4-space belongs to the class ΠN.

D. Example. The Moore plane is a locally Lindelöf, T_3-space and hence

it belongs to the class ΠN, but it is not a T_4-space.

E. <u>Definition</u>. Let X be a topological space. Then we define,

$$\|\mathcal{U}\| = \sup\{\,|U| : U \in \mathcal{U}\},$$

$$A(a,X) = \min\{k : a \text{ has a local } \pi\text{-base } \mathcal{U}_a \text{ with } \|\mathcal{U}_a\| \leq k\} + \aleph_0$$

$$A(D) = \sup\{A(a,X) : a \in D\},$$

$$A(X) = \min\{A(D) : \bar{D} = X\},$$

$$R(X) = \log A(X).$$

F. <u>Theorem</u>. Let X be a T_3 space. Then

$$|X| \leq 2^{wL(X)\chi(X)R(X)}.$$

<u>Proof</u>. Let $\alpha = wL(X)\chi(X)R(X)$ and $k = 2^{\alpha}$. Suppose $|X| > k$. Let $\mathcal{U}_a$ be an open neighbourhood base at a with $|\mathcal{U}_a| \leq \alpha$. Let $B \in P_{\leq\alpha}(X)$. Let $\mathcal{U}_B = \cup\{\mathcal{U}_a : a \in B\}$ and let $\mathcal{V}_B = \{\mathcal{U} \subseteq \mathcal{U}_B : |\mathcal{U}| \leq \alpha$ and $X - \overline{\cup\mathcal{U}_B} \neq \Phi\}$. We note that $|\mathcal{V}_B| \leq k$. Since $R(X) \leq \alpha$, there exists some $D \subset X$ such that $X = \bar{D}$ and $A(D) \leq k$. Hence for each $\mathcal{U} \in \mathcal{V}_B$ there exists a non-empty open subset $K(\mathcal{U})$ such that,

(i) $K(\mathcal{U}) \subseteq X - \overline{\cup\mathcal{U}}$

(ii) $|K(\mathcal{U})| \leq k$.

We shall define $G : P_{\leq\alpha}(X) \to P_{\leq k}(X)$ by $G(B) = \overline{B \cup (\cup\{K(\mathcal{U}) : \mathcal{U} \in \mathcal{V}_B\})}$.

Then, there exists some $A \subset X$ such that $|A| = k$ and $A \supseteq G(B)$ for every $B \in P_{\leq\alpha}(A)$. We claim that, $X = \overline{A^0}$. Suppose $X - \overline{A^0} \neq \Phi$.

Then since X is regular, there exists a point q and an open subset U in X such that $q \in U \subseteq \bar{U} \subseteq X - \overline{A^0}$. Let $V = X - \bar{U}$. Then $\overline{A^0} \subseteq V$ and since $RwL(\overline{A^0}) \leq \alpha$, there exists a $B \in P_{\leq\alpha}(\overline{A^0})$ such that,

$$\begin{aligned} \overline{A^0} &\subseteq \overline{\cup\{V_a : V_a \in \mathcal{U}_a,\ V_a \subseteq V \text{ and } a \in B\}} \\ &\subseteq \overline{V} \\ &\subseteq X - U. \end{aligned}$$

Now since $\partial(X) \leq \alpha$, A is a closed subset of X and hence $B \in P_{\leq\alpha}(A)]$.

Let $\mathcal{U} = \{V_a : V_a \in \mathcal{U}_a, V_a \subset V$ and $a \in B\}$; then $\mathcal{U} \in \mathcal{V}_B$ and hence we have $K(\mathcal{U}) \subset [G(B)]^0 \subseteq A^0 \subseteq \overline{A^0} \subseteq \overline{\cup\mathcal{U}}$. But $K(\mathcal{U}) \subseteq X - \overline{\cup\mathcal{U}}$. This is a contradiction. Hence it follows that $|X| \leq k$.

G. Corollary. If $X \in \Pi N$, then we have $|X| \leq 2^{wL(X)\chi(X)}$.

Proof. By E of 2.2, we note that $R(X) \leq wL(X)\chi(X)$. Now by the theorem,

$$\begin{aligned} |X| &\leq (2^{wL(X)\chi(X)})^{wL(X)\chi(X)} \\ &= 2^{wL(X)\chi(X)}. \end{aligned}$$

2.3. Normal T_1-Spaces

We shall define a new cardinal function $p\pi w(X)$ and we prove that for normal, T_1-spaces $|X| \leq p\pi w(X)^{wL(X)\psi(X)\partial(X)}$. We also obtain an upper bound for the number of compact subsets $K(X)$ in a normal, T_1-space by proving,

$$|K(X)| \le 2^{p\pi w(X)wL(X)} .$$

Let B be a dense subset of X such that $|B| \le d(X)$. Let $\mathcal{B} = \cup\{\mathcal{U}_p : p \in B\}$ where $\mathcal{U}_p$ is a local π-base at p. Then, clearly $\mathcal{B}$ is a π-base for X.

We shall study special open covers. An open cover $\mathcal{G}$ is said to be a <u>strong open cover</u> of X if it satisfies the following properties:

(i) For distinct points x and y in X, there exists a $G \in \mathcal{G}$ such that $x \in G$ and $y \notin G$.

(ii) If $B \in \mathcal{B}$ and $a \notin \bar{B}$, then there exists a $G \in \mathcal{G}$ such that $a \in G$ and $G \cap B = \Phi$, where $\mathcal{B}$ is the π-base above.

A. <u>Definition</u> (Charlesworth [9]). We define

$$p\pi w(X) = \min\{k : X \text{ has a strong open cover } \mathcal{G} \text{ such that each point of } X \text{ is in at most } k \text{ members of } \mathcal{G}\} + \aleph_0 .$$

In a T_3-space, $pws(X) \le p\pi w(X) \le pw(X)$.

B. <u>Proposition</u>. Let X be a set, let k be an infinite cardinal and suppose $\mathcal{G} \subseteq P(X)$ is such that each point of X is in at most k members of $\mathcal{G}$. If B is a subset of X, then the cardinality of the set of all finite minimal covers of B by elements of $\mathcal{G}$ does not exceed k.

(This is Miscenko's lemma.)

C. <u>Lemma</u>. Let X be a T_3-space. Then $|X| \le d(X)^{\psi(X)\partial(X)}$.

Proof. Let B be a dense subset of X such that $|B| \leq d(X)$. Let $k = \psi(X)\partial(X)$. Then, since $\partial(X) \leq k$, we can write $X = \cup\{\bar{T} : T \subseteq B, |T| \leq k\}$. But note that, $w(\bar{T}) \leq \rho(\bar{T})$ and by 2.2 of [9]

$$\begin{aligned} |\bar{T}| &\leq \rho(\bar{T})^{\psi(\bar{T})} \\ &\leq 2^{d(\bar{T})\psi(\bar{T})} \\ &\leq 2^k . \end{aligned}$$

It follows that
$$\begin{aligned} |X| &\leq 2^k \cdot |B|^k \\ &\leq d(X)^k \\ &= d(X)^{\psi(X)\partial(X)}. \end{aligned}$$

D. Lemma. Let X be a normal, T_1-space. Then $d(X) \leq p\pi w(X)^{wL(X)}$.

Proof. Let $k = p\pi w(X)$ and $\lambda = wL(X)$. We shall define, $G : [X]^{\leq k^\lambda} \to [X]^{k^\lambda}$. Let $A \in [X]^{\leq k^\lambda}$ and let $\mathcal{G}$ be a strong open cover of X such that each point of X is in at most k members of $\mathcal{G}$. We set $\mathcal{G}_A = \{G \in \mathcal{G} : G \cap A \neq \Phi\}$ and $\mathcal{M}_A = \{\mathcal{U} \in [\mathcal{G}_A]^{\leq\lambda} : X - \overline{\cup\mathcal{U}} \neq \Phi\}$. Now for each $\mathcal{U} \in \mathcal{M}_A$, choose $p(\mathcal{U}) \in X - \overline{\cup\mathcal{U}}$ and set $G(A) = A \cup (\cup\{p(\mathcal{U}) : \mathcal{U} \in \mathcal{M}_A\}$. Since $|\mathcal{M}_A| \leq k^\lambda$,

$$\begin{aligned} |G(A)| &\leq k^\lambda + k^\lambda \\ &= k^\lambda . \end{aligned}$$

Hence, $G(A) \in [X]^{k^\lambda}$ and this completes the construction of the set mapping G.

Now, by transfinite induction, we shall construct sets A_α for

$\alpha < \lambda^+$. Suppose we have defined A_β for $\beta < \alpha$; then we put $A_\alpha = G(\cup\{A_\beta : \beta < \alpha\})$. Clearly $A_\alpha \supseteq A_\beta$, if $\beta < \alpha$, and $|A_\alpha| \leq k^\lambda$. We set $A = \cup\{A_\alpha : \alpha \in \lambda^+\}$ and we claim that $X = \bar{A}$. Suppose $X - \bar{A} \neq \phi$; then we take a $q \in X - \bar{A}$. By the regularity of X, there exists a closed neighbourhood w of a such that $w \cap \bar{A} = \Phi$. Since X is normal, $RwL(\bar{A}) \leq \lambda$, and hence there exists a $T \in [\bar{A}]^{\leq\lambda}$ such that

(a) $\bar{A} \subseteq \overline{\cup\{G_y : y \in T \text{ and } G_y \in G\}}$, and

(b) $G_y \cap B = \Phi$ for every $y \in T$ where $B \in \mathcal{B}$, where $\mathcal{B}$ is the π-base constructed in the introduction of this section.

Let $\mathcal{U} = \{G_y : y \in T\}$. Then $X - \overline{\cup\mathcal{U}} \neq \Phi$. Since $G_y \cap \bar{A} \neq \Phi$, $G_y \cap A \neq \Phi$ and hence $G_y \cap A_{\alpha_y} \neq \Phi$ for some $\alpha_y \in \lambda^+$. let $\alpha = \sup\{\alpha_y : y \in T\}$. Since $|T| \leq \lambda$ and λ^+ is a regular cardinal, $\alpha < \lambda^+$. Also $G_y \cap A_\alpha \neq \Phi$ for every $y \in T$. Therefore $\mathcal{U} \in M_{A_\alpha}$ and hence $p(\mathcal{U}) \in G(A_\alpha) \subseteq A_{\alpha+1} \subseteq A$. But $p(\mathcal{U}) \in X - \overline{\cup\mathcal{U}} \subseteq X - \bar{A} \subseteq X - A$. This gives a contradiction. Hence we have $X = \bar{A}$, and thus the lemma.

E. Corollary. Let X be a normal T_1-space. Then $|X| \leq p\pi w(X)^{wL(X)\psi(X)\partial(X)}$.

Proof. Follows from lemmas C and D.

F. Theorem. Let X be a normal T_1-space. Then $K(X) \leq 2^{p\pi w(X)wL(X)}$.

Proof. Let $k = p\pi w(X)wL(X)$. Then by lemma D, there exists a strong open cover G such that,

(i) $|G| \leq 2^k$ and

(ii) every point x of X is in at most k members of $\mathcal{G}$.

Now we set

$$\mathcal{W} = \{U : U = \cup\mathcal{U} \text{ where } \mathcal{U} \in P_{< \aleph_0}(\mathcal{G})\}$$

and

$$\mathcal{B} = \{B : B = \cap\mathcal{V} \text{ where } \mathcal{V} \in P_{\leq k}(\mathcal{W})\}.$$

Then, we note that

$$|\mathcal{W}| \leq 2^k \qquad \text{and}$$

$$|\mathcal{B}| \leq (2^k)^k$$

$$= 2^k$$

We shall show that $\mathcal{K}(X) \subseteq \mathcal{B}$. Let $K \in \mathcal{K}(X)$. Then, since K is compact, we can find a $U \in \mathcal{W}$ such that $K \subseteq U$. Let $\mathcal{W}_K = \{U \in \mathcal{W} : K \subseteq U\}$. Now, by the proposition $|\mathcal{W}_K| \leq k$ and since $\mathcal{G}$ is point-separating, it follows that $\cap\mathcal{W}_K = K$. Hence $K \in \mathcal{B}$. Thus

$$|\mathcal{K}(X)| \leq |\mathcal{B}|$$

$$\leq 2^k.$$

This completes the proof of the theorem.

3. Relations Between the Lindelöf Number and the Weak Lindelöf Number

We introduce two cardinal functions, the feeble Lindolöf number $Q(X)$ and the turf number $T(X)$, and we show that,

(i) $Q(X) \leq wL(X)$

(ii) $L(X) \leq Q(X)T(X)$.

3.1. Background

Always $wL(X) \leq L(X)$ and in paracompact, T_2-spaces, $L(X) = wL(X)$.

Let A be a subset of X. If $\mathrm{Int}_X \mathrm{C\ell}_X A = \Phi$, we say that A is nowhere dense. If every nowhere dense subset of X is closed, then X is called a NODEC space.

We recall that $p(X)$ is the closed spread of X. In the class of NODEC spaces, the following can easily be established:

$$L(X) = p(X)wL(X).$$

In the next section we shall study the possibilities of such relations in arbitrary topological spaces.

3.2. Main Theorem

We study some properties of $Q(X)$ and $T(X)$ and we show that $L(X) \leq Q(X)T(X)$.

A. Definition. A collection $\{\mathcal{V}_\alpha : \alpha \in I\}$ of locally finite open covers of a topological space X is called a turf for X if the following hold for each $a \in X$:

(i) $L(a) = \cap\{\cap\{V : a \in V \in \mathcal{V}_\alpha\} : \alpha \in I\}$ is Lindelöf,

(ii) if $L(a)$ is contained in an open subset U of X, then there exists a finite subset F of I such that

$$\begin{aligned} L(a) &\subseteq \cap\{\cap\{V \in \mathcal{V}_\alpha : a \in V\} : \alpha \in I\} \\ &\subseteq U. \end{aligned}$$

If X is a regular space, and $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ is a base for X, then the collection $\mathcal{V}_{\alpha,\beta} = \{X - \bar{B}_\alpha, B_\beta\}$ where $(\alpha,\beta) \in I$ and $I = \{(\alpha,\beta) \in A \times A : \bar{B}_\alpha \subset B_\beta\}$ is a turf for X. Thus all regular spaces have turfs.

B. Definition. Let X be a regular space. Then the turf number $T(X)$ of X is defined by,

$$T(X) = \min\{k : X \text{ has a turf } \{\mathcal{V}_\alpha : \alpha \in I\} \text{ with } |I| = k\} + \omega .$$

It is clear from A that, $T(X) \leq w(X)$. For a discrete space D, $T(D) = \aleph_0$, $w(D) = |D|$.

C. Examples. Let $\mathbb{R}$ denote the real line, E the Sorgenfrey line and M the Moore plane. Let $\Omega_0 = [0,\omega_1)$ be the open, uncountable ordinal space. Then,

(i) $T(\mathbb{R}) = T(E) = \aleph_0$

(ii) $T(M) = c$

(iii) $T(\Omega_0) = \omega_1$

(iv) $T(E \times E) = c$

D. Definition. The pseudo-compactness number $pc(X)$ of X is defined by,

$$pc(X) = \sup \{|G| : G \text{ is a locally finite collection of non-empty open subsets of } X\} .$$

In this terminology, a topological space X is pseudo-compact if and only if $pc(X) < \aleph_0$.

It is clear that $pc(X) \leq L(X)$. This does not guarantee any relationship between Lindelöf spaces and pseudo-compact spaces. The two classes are just different as we know by well known examples $\Omega_0 = [0,\omega_1)$ and $\mathbb{N}$.

We define the feeble Lindelöf number $Q(X)$ as,

$$Q(X) = pc(X) + \aleph_0 .$$

E. Example. Let $X = \beta\mathbb{R} - (\beta\mathbb{N} - \mathbb{N})$. Then $Q(X) = \aleph_0$ and $L(X) = \aleph_1$. We shall show that every weakly Lindelöf space is feebly Lindelöf.

F. Lemma. Let X be any topological space. Suppose X is weakly $m^+ - m^+$ compact; then $Q(X) \leq m$.

Proof. Suppose $Q(X) > m$. Then there exists a locally-finite collection

$\mathcal{G}$ of non-empty open subsets of X such that $|\mathcal{G}| = m^+$. Let $\mathcal{G} = \{G_i : i \in m^+\}$ and, for each $\alpha < m^+$, $U = X - \cup\{\bar{G}_i : i \geq \alpha\}$. Then $\{U_\alpha : \alpha \in m^+\}$ is an open cover of X and since $\{U_\alpha : \alpha \in m^+\}$ is an increasing collection and m^+ is a regular cardinal, it follows that X cannot be weakly $m^+ - m^+$ compact.

G. Corollary. $Q(X) \leq wL(X)$.

In particular, $Q(X) \leq c(X)$.

H. Theorem. Let X be a regular space. Then $L(X) \leq Q(X)T(X)$.

Proof. Let $\alpha = Q(X)T(X)$. Let $\{\mathcal{V}_\alpha : \alpha \in I\}$ be a turf for X wich $|I| \leq \alpha$ and $|\mathcal{V}_\alpha| \leq \alpha$ for every $\alpha \in I$. Let $\mathcal{G}$ be an open cover of X and let $\mathcal{H}$ be the collection of all countable unions of elements of $\mathcal{G}$. We set $\mathcal{V} = \{V : V = \cap \mathcal{B}$ where $\mathcal{B} \in \mathcal{P}_{<\aleph_0}(\cup\mathcal{V}_\alpha : \alpha \in I)\}$ and $\mathcal{V}' = \{V \in \mathcal{V} : V \subseteq H$ for some $H \in \mathcal{H}\}$. We note that $|\mathcal{V}| \leq \alpha$ and hence $|\mathcal{V}'| \leq \alpha$. We shall prove that $X = \cup\mathcal{V}'$; from this it follows that $L(X) \leq \alpha$. To prove $X = \cup\mathcal{V}'$, let $a \in X$; then, since $L(a)$ is Lindelöf, there exists an $H \in \mathcal{H}$ such that $L(a) \subseteq H$ where H is an open subset of X. Hence there exists $F \in P_{<\aleph_0}(I)$ such that $L(a) \subseteq \cap\{\cap\{V \in \mathcal{V}_\alpha : a \in \mathcal{V}\} : \alpha \in F\} \subseteq H$. Let $K = \cap\{\cap\{V \in \mathcal{V}_\alpha : a \in V\} : \alpha \in F\}$. Then we note that $a \in K \in \mathcal{V}'$.

I. Corollary. Let X be a regular space. Then $L(X) \leq wL(X)T(X)$.

REFERENCES AND REMARKS

Chapter I.

No new results are contained in this chapter. The material is introductory and can be found in standard references for the most part.

Section 1 contains an outline of basic facts about cardinal arithmetic.

Section 2 is our introduction to cardinal invariants, particularly $\chi(X)$, $\psi(X)$ and $L(X)$. Historical facts about cardinal invariants can be found in [1] and [3].

In Section 3, we study m-n filters, which were introduced and investigated by J.E. Vaughan in 1972.

Section 4 contains important theorems about generalized products; this material can be found in Comfort and Negrepontis [12].

Chapter II.

1.1G appears to be new. 1.2A and 1.2B can be found in Noble [53]. 2.2G appears to be new. 2.3A is Exercise 20E in Willard [80]. 2.3G, 31.B, 3.1G, 4.1F and 4.2C all seem to be new.

The example based on fans given by M.G. Tkacenko (Topology and Applications 15(1983), 93-98) shows that Theorem 4.1F stands in its best possible form for regular spaces.

Chapter III.

1.1F, 1.1G, 1.2A and 1.2G appear to be new. 2.1G is analogous to Vaughan's lemma 3.1 [73]. 2.1H is new, but the techniques used are to be credited to Gal [18]. The countable version of 2.2C is due to Hajeek and

Todd [29]. All of 3.2B, 32.E, .3.3A, 4.1E and 4.2A seem to be new. 4.3B appears as 1.3 in Ulmer's [69]. Although 4.3C, 4.3D and 4.3G may be new, similar theorems for m-n compactness can be found in Vaughan [75].

Theorem 3.3A unifies Theorems 4.6 and 4.7 on feeble compactness in Scarborough and Stone [62].

Chapter IV.

The proof of 1.1F given here is simpler than that given by Juhasz (cf. 2.27 of [34]). 1.1H is an improvement of Theorem 2.5 in [34]. 1.2B uses well-known closure techniques of Saprivoski. 1.2C appears to be new, as does 1.2G, but the techniques used in the latter are drawn from 3.29 in Juhasz' [34]. 1.2I is a new observation.

The material of section 1.3 appears to be new. The material of section 2.2 likewise, except that the setup in the proof of Theorem F is similar to that of Juhasz 2.36 [34]. Section 2.3 appears to be new.

3.2A is based on Arhangelśkii's famous p-spaces and an observation of Hodel and Burke about "pluming degree". 3.2I seems to be new; see also 4.3 of [4].

Example 1.3E shows that theorem 1.2C provides a better bound than Arhangelśkii's famous $|X| \leq x^{\chi(X)L(X)}$ (1970) and also differs from Hajnal and Juhasz $|X| \leq 2^{\chi(X)c(X)}$ (1976).

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AN ADDED NOTE

In 2.18 of [32], I. Juhãsz has shown that $K(X) \leq [\chi(X)]^+$ for connected spaces and noted that the result cannot be improved even in the class of paracompact, connected spaces. In contrast to this, the following theorem will show that in the class of weakly Lindelöf spaces $K(X) \leq \chi(X)$:

Theorem Let X be a topological space. Then $K(X) \leq wL(X)\partial(X)$ ·